

DOUBLE QUINTIC SYMMETROIDS, REYE CONGRUENCES, AND THEIR DERIVED EQUIVALENCE

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Dedicated to Professor Yujiro Kawamata on the occasion of his 60th birthday

ABSTRACT. Let \mathcal{Y} be the double cover of the quintic symmetric determinantal hypersurface in \mathbb{P}^{14} . We consider Calabi-Yau threefolds Y defined as smooth linear sections of \mathcal{Y} . In our previous works, we have shown that these Calabi-Yau threefolds Y are naturally paired with Reye congruence Calabi-Yau threefolds X by the projective duality of \mathcal{Y} , and X and Y have several interesting properties from the view point of mirror symmetry and projective geometry. In this paper, we prove the derived equivalence between a linear section Y of \mathcal{Y} and the corresponding Reye congruence Calabi-Yau threefold X .

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1. INTRODUCTION

Non-birational smooth Calabi-Yau threefolds which have an equivalent derived category are of considerable interest from the view point of the homological mirror symmetry due to Kontsevich [Ko]. As such an example, it has been proved in [BC] that smooth Calabi-Yau threefolds X and Y which are given respective smooth linear sections of $G(2, 7)$ and $\text{Pfaff}(7)$ (see [Ro]) are derived equivalent: $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$. To our best knowledge, this pair is the first example of derived equivalent, but non-birational smooth Calabi-Yau threefolds with Picard number one. In [Ku2], the derived equivalence has been understood as a corollary of a more general statement that a non-commutative resolution of $\text{Pfaff}(7)$ is homologically projective dual to $G(2, 7)$. The homological projective duality is a general framework which has been proposed in [Ku1] to generalize the classical projective duality in the theory of derived category. In the case of $G(2, 7)$ and $\text{Pfaff}(7)$, the classical projective geometries involved are that of the projective space of skew symmetric matrices $\mathbb{P}(\wedge^2 \mathbb{C}^7)$ and its dual projective space $\mathbb{P}(\wedge^2 (\mathbb{C}^*)^7)$.

In the previous work [HoTa1], by studying mirror symmetry of the Calabi-Yau threefold X of a Reye congruence, we encountered a similar phenomenon as above within the projective space of symmetric matrices $\mathbb{P}(\text{S}^2 \mathbb{C}^5)$ and its dual projective space $\mathbb{P}(\text{S}^2 (\mathbb{C}^*)^5)$. We have observed that X should be paired with another smooth Calabi-Yau threefold Y , which is given as a linear section of the double quintic symmetroid orthogonal to X in the projective duality. We are also led to the prediction [ibid. Conj.2] of their derived equivalence. The main result of this paper is to show this prediction is affirmative.

Let $V = \mathbb{C}^5$ and define X as a smooth linear section of the second symmetric product of $\mathcal{X} = \text{S}^2 \mathbb{P}(V)$ in $\mathbb{P}(\text{S}^2 V)$. Then Y is given by the orthogonal linear section of the double symmetroid \mathcal{Y} , which is the double cover of the determinantal symmetroid \mathcal{H} in $\mathbb{P}(\text{S}^2 V^*)$. \mathcal{X} has a natural resolution $\tilde{\mathcal{X}}$ given by the Hilbert scheme of two points. As for \mathcal{Y} , a nice desingularization $\widetilde{\mathcal{Y}}$ has been obtained in our recent paper [HoTa3]. Furthermore, it has been found that a finite collection of sheaves $(\mathcal{F}_i)_{i \in I}$ on $\tilde{\mathcal{X}}$ introduces a dual Lefschetz collection in the derived categories $\mathcal{D}^b(\tilde{\mathcal{X}})$ [ibid. Thm.3.4.4], and correspondingly a collection of sheaves $(\mathcal{E}_i)_{i \in I}$ on $\widetilde{\mathcal{Y}}$ defines a Lefschetz collection in $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ [ibid. Thm.8.1.1]. In this paper, we focus on a certain closed subscheme Δ in $\widetilde{\mathcal{Y}} \times \tilde{\mathcal{X}}$ and construct a locally free resolution of

its ideal sheaf \mathcal{I} in terms of these sheaves (Theorem 6.1.1). Considering a Fourier-Mukai functor with its kernel I being the restriction of the sheaf \mathcal{I} to $Y \times X$ in $\widetilde{\mathcal{Y}} \times \mathcal{X}$, we prove the derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ (**Theorem 9.0.2**).

We introduce the subscheme Δ from the flag variety $\Delta_0 := F(2, 3, V)$ in $G(3, V) \times G(2, V)$. As we summarize in Subsection 2.2, there is a morphism from the Hilbert scheme \mathcal{X} of two points on $\mathbb{P}(V)$ to the Grassmannian $G(2, V)$. In contrast to this, the geometry of the double quintic symmetroid \mathcal{Y} is more involved. It contains, however, interesting geometry of quadrics in $\mathbb{P}(V)$, more precisely, it turns out that \mathcal{Y} describes the connected families of planes contained in singular quadrics, which are represented by conics on $G(3, V)$ for rank 4 quadrics (Subsection 2.3 and Section 4). In particular, we see that there is a generically conic bundle $\mathcal{Z} \rightarrow \mathcal{Y}$, where \mathcal{Z} parameterizes pairs of singular quadrics and planes therein, and hence there exists a natural morphism $\mathcal{Z} \rightarrow G(3, V)$. Roughly speaking, the subscheme Δ is constructed by pulling back Δ_0 by the morphism $\mathcal{Z} \times \mathcal{X} \rightarrow G(3, V) \times G(2, V)$, pushing forward by the morphism $\mathcal{Z} \times \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{X}$ and taking the transform by the birational morphism $\widetilde{\mathcal{Y}} \times \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{X}$. The observation that the families of planes in rank 4 quadrics are represented by conics on $G(3, V)$ implies that \mathcal{Y} is birational to the Hilbert scheme of conics on $G(3, V)$ studied by [IM]. This opens the way to describe the birational geometry of \mathcal{Y} (see (4.1)).

The choice of the subscheme Δ comes from our observation on the so-called BPS numbers of Y listed in [HoTa1, Table 3]. Let I_x be the restriction of the ideal sheaf I to $Y \times \{x\}$. We will show that I_x defines a curve C_x on Y of arithmetic genus 3 and degree 5 parameterized by X , and C_x is smooth if X and x are general (Propositions 3.1.2 and 8.2.1). Our observation/discovery about this family of curves is that this can be identified in the table of BPS numbers $n_g^Y(d)$ at genus 3 and degree 5 as

$$n_3^Y(5) = 100 = (-1)^{\dim X} e(X) \times 2,$$

where $e(X) = -50$ is the Euler number of X . We note that the signed Euler number $(-1)^{\dim X} e(X)$ is in accord with the counting rule of the BPS numbers for a family of curves [GV]. The factor 2 indicates that there should be another family of curves parameterized by X . Interestingly, we find that there is a “shadow” curve of the same genus and degree for each C_x (Fig.1 in Subsection 3.1), which explains this factor. The BPS numbers are integral numbers which we calculate by using mirror symmetry, and its mathematical ground is still missing in general. We believe, however, that our observation above may be justified by the Donaldson-Thomas (DT) invariants or the Pandharipande-Thomas (PT) invariants associated with a suitable moduli problem of ideal sheaves or stable pairs (see [PT] and reference therein). It is expected that the Calabi-Yau threefold X arises as a suitable moduli space of the ideal sheaf of curves on Y and the derived equivalence between the two is a consequence of this.

Here we should remark some similarity of our construction to that of the Grassmann-Pfaffian case. The proof of derived equivalence due to [BC] (and also [Ku2]) is based on a certain incidence relation on $G(2, 7) \times \text{Pfaff}(7)$ which gives rise the kernel of a Fourier-Mukai functor. Our proof starting with Δ_0 is basically parallel to this, although the formulation of our incidence relation Δ and the corresponding ideal sheaf are much more involved and requires the desingularization $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ [HoTa3] which follows from detailed investigations of the birational geometry of \mathcal{Y} . In the

Grassmann-Pfaffian case, the restriction I_y to $\{y\} \times X$ of the ideal sheaf I defines a generically smooth family of curves of genus 6 and degree 14 on X . Thus, in both cases, we observe that suitable families of curves play important roles for the proofs of the derived equivalence. This seems to indicate some relation in general between the derived equivalence and the suitable moduli problem such as DT/PT invariants, as addressed above. In the Grassmann-Pfaffian case, however, we read the corresponding BPS as $n_6^X(14) = 123,676$ (see [HoTa1, (4.2)]) and there might be some complications in the possible moduli interpretation in terms of DT/PT invariants.

Here we outline the present paper. In Section 2, we summarize some basic results on which our arguments rely. In Section 3, we construct a family of curves on Y parameterized by X (Proposition 3.1.2) and show that it comes with another family of “shadow” curves. Also we calculate the Brauer group of Y as a corollary. In Section 4, based on the results [HoTa3], we summarize the birational geometry of \mathcal{Y} and introduce the desingularization $\widetilde{\mathcal{Y}}$. In Section 5, we study the birational geometry of the universal family of conics on $G(3, V)$. In Section 6, we introduce the subscheme Δ representing the incidence relation on $\widetilde{\mathcal{Y}} \times \mathcal{X}$ and its ideal sheaf \mathcal{I} . Dividing the process into four major steps, we obtain a locally free resolution of \mathcal{I} in terms of the collections of locally free sheaves $(\mathcal{E}_i)_{i \in I}$ and $(\mathcal{F}_i)_{i \in I}$ (Theorem 6.1.1). In section 7, we show that the subscheme Δ is contained in (the pullback of) the universal family of hyperplane sections \mathcal{V} . In Section 8, restricting the ideal sheaf to $Y \times X$, we show that this defines a family of curve on Y which is flat over X , and coincides with the family obtained in Section 3 (Proposition 8.2.1). In Section 9, using [HoTa3, Thm.8.1.1], which claim the vanishing of Ext groups among the collections $(\mathcal{E}_i)_{i \in I}$ and $(\mathcal{F}_i)_{i \in I}$, we show the derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$. Section 10 is devoted to discussions of related subjects.

Acknowledgements: This paper is supported in part by Grant-in Aid Scientific Research (C 18540014, S.H.) and Grant-in Aid for Young Scientists (B 20740005, H.T.).

Glossary of notation.

We often abuse Cartier divisors and invertible sheaves.

$\mathbb{P}(W)$: the projectivization of a vector space W .

$\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}(W)}(1)|_X$ if a variety X is naturally embedded in $\mathbb{P}(W)$.

$\mathbb{P}(\mathcal{E})$: the projectivization of a locally free sheaf \mathcal{E} on a variety X .

$H_{\mathbb{P}(\mathcal{E})}$: the tautological divisor of $\mathbb{P}(\mathcal{E})$.

$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$: the tautological invertible sheaf of $\mathbb{P}(\mathcal{E})$.

$G(r, \mathcal{E})$: the Grassmann bundle which parameterizes $r - 1$ -dimensional linear subspaces in fibers of $\mathbb{P}(\mathcal{E})$.

V : a (fixed) 5 dimensional complex vector space. $\mathbb{P}^4 := \mathbb{P}(V)$.

V_i : an i -dimensional vector subspace of V .

$F(a, b, V) := \{(V_a, V_b) \mid V_a \subset V_b \subset V\}$ (the flag variety).

We also use the following notation which simplifies lengthy formulas:

$\Omega(1) := \Omega_{\mathbb{P}(V)}(1)$.

$\Omega(1)^{\wedge i} := \wedge^i(\Omega_{\mathbb{P}(V)}(1))$ for $i \geq 2$.

$T(-1) := T_{\mathbb{P}(V)}(-1)$.

$T(-1)^{\wedge i} := \wedge^i(T(-1))$ for $i \geq 2$.

$\mathcal{O}(i) := \mathcal{O}_{\mathbb{P}(V)}(i)$ for $i \in \mathbb{Z}$.

2. PRELIMINARIES

Here we summarize the basic results on which our proof of the derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ rely. We also summarize the construction of the Calabi-Yau threefolds X and Y which has been described in [HoTa1, HoTa3].

2.1. Basic general results.

For the computations of cohomology groups which appear in this paper, we use the Bott theorem about the cohomology groups of Grassmann bundles below extensively.

For a locally free sheaf \mathcal{E} of rank r on a variety and a nonincreasing sequence $\beta = (\beta_1, \beta_2, \dots, \beta_r)$ of integers, we denote by $\Sigma^\beta \mathcal{E}$ the associated locally free sheaf with the Schur functor Σ^β .

Theorem 2.1.1 (Bott theorem). *Let $\pi: G(r, \mathcal{A}) \rightarrow X$ be a Grassmann bundle for a locally free sheaf \mathcal{A} on a variety X of rank n and $0 \rightarrow \mathcal{S}^* \rightarrow \mathcal{A} \rightarrow \mathcal{Q} \rightarrow 0$ the universal exact sequence. For $\beta := (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$ ($\alpha_1 \geq \dots \geq \alpha_r$) and $\gamma := (\alpha_{r+1}, \dots, \alpha_n) \in \mathbb{Z}^{n-r}$ ($\alpha_{r+1} \geq \dots \geq \alpha_n$), we set $\alpha := (\beta, \gamma)$ and $\mathcal{V}(\alpha) := \Sigma^\beta \mathcal{S} \otimes \Sigma^\gamma \mathcal{Q}^*$. Finally, let $\rho := (n, \dots, 1)$, and, for an element σ of the n -th symmetric group \mathfrak{S}_n , we set $\sigma^\bullet(\alpha) := \sigma(\alpha + \rho) - \rho$.*

- (1) *If $\sigma(\alpha + \rho)$ contains two equal integers, then $R^i \pi_* \mathcal{V}(\alpha) = 0$ for any $i \geq 0$.*
- (2) *If there exists an element $\sigma \in \mathfrak{S}_n$ such that $\sigma(\alpha + \rho)$ is strictly decreasing, then $R^i \pi_* \mathcal{V}(\alpha) = 0$ for any $i \geq 0$ except $R^{l(\sigma)} \pi_* \mathcal{V}(\alpha) = \Sigma^{\sigma^\bullet(\alpha)} \mathcal{A}^*$, where $l(\sigma)$ represents the length of $\sigma \in \mathfrak{S}_n$.*

Proof. See [Bo], [D], or [W, (4.19) Corollary]. \square

Theorem 2.1.2 (Grothendieck-Verdier duality). *Let $f: X \rightarrow Y$ be a proper morphism of smooth varieties X and Y . Set $n := \dim X - \dim Y$. We have the following functorial isomorphism:*

For $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ and $\mathcal{E}^\bullet \in \mathcal{D}^b(Y)$,

$$Rf_* R\mathcal{H}om(\mathcal{F}^\bullet, Lf^* \mathcal{E}^\bullet \otimes \omega_{X/Y}[n]) \simeq R\mathcal{H}om(Rf_* \mathcal{F}^\bullet, \mathcal{E}^\bullet).$$

In particular, if \mathcal{E}^\bullet and \mathcal{F}^\bullet are locally free (and then we write them simply \mathcal{E} and \mathcal{F}) and if $R^\bullet f_ \mathcal{F} = 0$, then*

$$R^{\bullet+n} f_*(\mathcal{F}^* \otimes \mathcal{E} \otimes \omega_{X/Y}) \simeq \mathcal{E}xt^\bullet(f_* \mathcal{F}, \mathcal{E}).$$

Proof. See [Huy, Theorem 3.34]. \square

Throughout this paper, we only need one result from the theory of derived category as follows:

Theorem 2.1.3. *Let X and Y be smooth projective varieties and \mathcal{P} a coherent sheaf on $X \times Y$ flat over X . Then the Fourier-Mukai transform $\Phi_{\mathcal{P}}: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ (\mathcal{P} is called the kernel for $\Phi_{\mathcal{P}}$) is fully faithful if and only if the following two conditions are satisfied:*

- (i) *For any point $x \in X$, it holds $\mathrm{Hom}(\mathcal{P}_x, \mathcal{P}_x) \simeq \mathbb{C}$, and*
- (ii) *if $x_1 \neq x_2$, then $\mathrm{Ext}^i(\mathcal{P}_{x_1}, \mathcal{P}_{x_2}) = 0$ for any i .*

Moreover, under these conditions, $\Phi_{\mathcal{P}}$ is an equivalence of triangulated categories if and only if $\dim X = \dim Y$ and $\mathcal{P} \otimes \mathrm{pr}_1^ \omega_X \simeq \mathcal{P} \otimes \mathrm{pr}_2^* \omega_Y$.*

In particular, if $\dim X = \dim Y$, $\omega_X \simeq \mathcal{O}_X$ and $\omega_Y \simeq \mathcal{O}_Y$, then $\Phi_{\mathcal{P}}$ is fully faithful if and only if it is an equivalence.

Proof. See [BO, Theorem 1.1], [B, Theorem 1.1], [Huy, Corollary 7.5 and Proposition 7.6]. \square

In this paper, we adopt the following definition of Calabi-Yau variety and also Calabi-Yau manifold.

Definition 2.1.4. We say a normal projective variety X a *Calabi-Yau variety* if X has only Gorenstein canonical singularities, the canonical bundle of X is trivial, and $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$. If X is smooth, then X is called a *Calabi-Yau manifold*. A smooth Calabi-Yau threefold is abbreviated as a Calabi-Yau threefold. \square

2.2. The Hilbert scheme $\tilde{\mathcal{X}}$ of two points on $\mathbb{P}(V)$.

Let \mathcal{X} be the Chow variety of two points on $\mathbb{P}(V)$ embedded by the Chow form into $\mathbb{P}(S^2 V)$. Denote by \mathcal{X}_0 the second Veronese variety $v_2(\mathbb{P}(V))$. It is a well-known fact that $\mathcal{X}_0 = \mathrm{Sing} \mathcal{X}$ and \mathcal{X} is the secant variety of \mathcal{X}_0 . If we take a coordinate of $S^2 V$ so that it represents a generic 5×5 symmetric matrix, then \mathcal{X}_0 (resp. \mathcal{X}) is characterized as the locus of rank 1 (resp. rank ≤ 2) symmetric matrices.

Let $\tilde{\mathcal{X}}$ be the Hilbert scheme of 0-dimensional subschemes of length two in $\mathbb{P}(V)$, which will be called the Hilbert scheme of two points in $\mathbb{P}(V)$ hereafter. A 0-dimensional subscheme of length two may be determined from the corresponding

0-cycle η of length two on $\mathbb{P}(V)$ and a line $l \subset \mathbb{P}(V)$ containing η , and vice versa. Hence, we have an isomorphism $\tilde{\mathcal{X}} \simeq \mathbb{P}(\mathcal{S}^2 \mathcal{F}^*)$, where \mathcal{F} is the dual of the universal subbundle of rank two on $G(2, V)$. Let

$$(2.1) \quad 0 \rightarrow \mathcal{F}^* \rightarrow V \otimes \mathcal{O}_{G(2, V)} \rightarrow \mathcal{G} \rightarrow 0$$

be the universal exact sequence on $G(2, V)$. By the induced injection $\mathcal{S}^2 \mathcal{F}^* \hookrightarrow \mathcal{S}^2 V \otimes \mathcal{O}_{G(2, V)}$, we obtain a morphism $\tilde{\mathcal{X}} \rightarrow \mathbb{P}(\mathcal{S}^2 V) \times G(2, V) \rightarrow \mathbb{P}(\mathcal{S}^2 V)$. Then the tautological divisor of $\mathbb{P}(\mathcal{S}^2 \mathcal{F}^*)$ is the pull-back of $\mathcal{O}_{\mathbb{P}(\mathcal{S}^2 V)}(1)$. The image of this morphism is nothing but \mathcal{X} and the induced morphism $f: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ coincides with the Hilbert-Chow morphism. Moreover, f is the blow-up along \mathcal{X}_0 .

$$\begin{array}{ccc} & \tilde{\mathcal{X}} & \\ f \swarrow & & \searrow g \\ \mathcal{X} & & G(2, V). \end{array}$$

We define the following divisors on $\tilde{\mathcal{X}}$:

$$H_{\tilde{\mathcal{X}}} = f^* \mathcal{O}_{\mathcal{X}}(1) \text{ and } L_{\tilde{\mathcal{X}}} = g^* \mathcal{O}_{G(2, V)}(1).$$

We also denote by E_f the f -exceptional divisor.

2.3. The double quintic symmetroid \mathcal{Y} .

Hereafter we assume $V \simeq \mathbb{C}^5$ and write by V^* the dual vector space of V . Let $\mathcal{H} \subset \mathbb{P}(\mathcal{S}^2 V^*)$ be the locus of singular quadrics in $\mathbb{P}(V)$, which will be called the (generic) *quintic symmetroid* since it is the hypersurface defined by the determinant of the (generic) 5×5 symmetric matrix. It is the locus of 5×5 symmetric matrices of rank ≤ 4 .

Let $\mathcal{U} := \{(t, [Q]) \mid t \in \text{Sing } Q\} \subset \mathbb{P}(V) \times \mathbb{P}(\mathcal{S}^2 V^*)$. Then the natural morphism $p: \mathcal{U} \rightarrow \mathcal{H}$ is a desingularization of \mathcal{H} (see [HoTa3, Subsect.4.1]).

To construct the double cover \mathcal{Y} of \mathcal{H} branched along the locus of symmetric matrices of rank ≤ 3 , we introduce the variety \mathcal{Z} which parameterizes the pair of quadrics Q in $\mathbb{P}(V)$ and planes $\mathbb{P}(\Pi)$ such that $\mathbb{P}(\Pi) \subset Q$. To describe \mathcal{Z} more explicitly, consider the universal exact sequence on $G(3, V)$;

$$(2.2) \quad 0 \rightarrow \mathcal{U}^* \rightarrow V \otimes \mathcal{O}_{G(3, V)} \rightarrow \mathcal{W} \rightarrow 0.$$

The surjection $\mathcal{S}^2 V^* \otimes \mathcal{O}_{G(3, V)} \rightarrow \mathcal{S}^2 \mathcal{U}$ follows from the dual sequence. Then we define a locally free sheaf \mathcal{E}^* on $G(3, V)$ by

$$(2.3) \quad 0 \rightarrow \mathcal{E}^* \rightarrow \mathcal{S}^2 V^* \otimes \mathcal{O}_{G(3, V)} \rightarrow \mathcal{S}^2 \mathcal{U} \rightarrow 0.$$

Then, since the fiber of $\mathcal{S}^2 \mathcal{U}$ over a point $[\Pi] \in G(3, V)$ may be identified with the quadrics on $\mathbb{P}(\Pi)$, the fiber of the kernel \mathcal{E}^* represents the quadrics which contain the plane $\mathbb{P}(\Pi)$. Namely we have

$$\mathcal{Z} = \mathbb{P}(\mathcal{E}^*) \subset G(3, V) \times \mathbb{P}(\mathcal{S}^2 V^*).$$

Note that the image of the naturally induced morphism $\mathcal{Z} \rightarrow \mathbb{P}(\mathcal{S}^2 V^*)$ coincides with the singular quadrics \mathcal{H} , since a smooth quadric does not contain a plane.

Let $\mathcal{Z} \xrightarrow{\pi_{\mathcal{Z}}} \mathcal{Y} \xrightarrow{\rho_{\mathcal{Y}}} \mathcal{H}$ be the Stein factorization of the natural morphism $\mathcal{Z} \rightarrow \mathcal{H}$. Then $\rho_{\mathcal{Y}}: \mathcal{Y} \rightarrow \mathcal{H}$ is the finite double covering branched along the locus of quadrics of rank less than or equal to three. \mathcal{Y} is called the (generic) *double quintic*

symmetroid. We say that $y \in \mathcal{Y}$ is a *rank i point* if $\rho_y(y) \in \mathcal{H}$ corresponds to a quadric of rank i . $G_{\mathcal{Y}} := \text{Sing } \mathcal{Y}$ is the subset consisting of rank 1 and 2 points ([HoTa3, Prop.6.9.2]). We introduce divisors on \mathcal{Y} and \mathcal{Z} , respectively, by

$$M_{\mathcal{Y}} = \rho_{\mathcal{Y}}^* \mathcal{O}_{\mathcal{H}}(1) \text{ and } M_{\mathcal{Z}} = \pi_{\mathcal{Z}}^* \circ \rho_{\mathcal{Y}}^* \mathcal{O}_{\mathcal{H}}(1),$$

where $\mathcal{O}_{\mathcal{H}}(1) := \mathcal{O}_{\mathbb{P}(\mathbb{S}^2 V^*)}(1)|_{\mathcal{H}}$.

Consider the fiber $\mathcal{Z}_{[Q]}$ of the morphism $\mathcal{Z} \rightarrow \mathcal{H}$ over a quadric $[Q] \in \mathcal{H}$. If $\text{rank } Q = 4$, then Q is a cone over the smooth quadric ($\simeq \mathbb{P}^1 \times \mathbb{P}^1$) in $\mathbb{P}(V/V_1)$ with the vertex $[V_1]$, and the planes in Q consist of two different \mathbb{P}^1 -families which correspond to the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. If $\text{rank } Q = 3$, then Q is a cone over the smooth quadric in $\mathbb{P}(V/V_2)$ with the vertex $\mathbb{P}(V_2) \simeq \mathbb{P}^1$, and in this case there is only one \mathbb{P}^1 -family of planes in Q . Thus the morphism $\pi_{\mathcal{Z}}: \mathcal{Z} \rightarrow \mathcal{Y}$ of the Stein factorization is a generically conic bundle; generic points of \mathcal{Y} are represented by pairs (Q, q) of quadrics Q of rank 4 (or 3) and connected families q of planes in Q , where q represent conics in $G(3, V)$ ([HoTa3, Prop.4.2.5]). It turns out that several birational models of $\mathcal{Z} \rightarrow \mathcal{Y}$ play crucial roles to construct the kernel of a Fourier-Mukai functor giving an equivalence of $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$.

Based on this observation, the birational geometry of \mathcal{Y} has been studied in terms of the Hilbert scheme \mathcal{H}_0 of conics on $G(3, V)$ [ibid. Sect.5 and Sect.6].

The following computations of the Chern classes of the locally free sheaf \mathcal{E} on \mathcal{Z} will be used in the next section.

Lemma 2.3.1. $c_1(\mathcal{E}) = c_1(\mathcal{O}_{G(3, V)}(4))$ and $c_2(\mathcal{E}) = 5c_2(\mathcal{W}) + 6c_1(\mathcal{O}_{G(3, V)}(1))^2$.

Proof. This follows from standard computations of Chern classes by using the exact sequences (2.2) and (2.3). Note that $c_1(\mathcal{O}_{G(3, V)}(1))$ is given by the Schubert cycle σ_1 , which is $c_1(\mathcal{U})$ in our notation. Since $\text{rk } \mathcal{U} = 3$, the first relation follows from $c_1(\mathcal{E}) = c_1(\mathbb{S}^2 \mathcal{U}) = 4c_1(\mathcal{U})$. For the second relation, we derive $c_2(\mathbb{S}^2 \mathcal{U}) = 5c_1(\mathcal{U})^2 + 5c_2(\mathcal{U})$ and use $c_2(\mathcal{W}) = c_1(\mathcal{U})^2 - c_2(\mathcal{U})$, $c_2(\mathcal{E}) = c_1(\mathbb{S}^2 \mathcal{U})^2 - c_2(\mathbb{S}^2 \mathcal{U})$. \square

2.4. Calabi-Yau threefolds X and Y .

Let us identify $\mathbb{P}(\mathbb{S}^2 V^*)$ with the space of quadrics in $\mathbb{P}(V)$. Then a 4-plane $P \subset \mathbb{P}(\mathbb{S}^2 V^*)$ may be regarded as a 4-dimensional linear system of quadrics $P = |Q_1, Q_2, \dots, Q_5|$ with the quadrics Q_i defined by the corresponding 5×5 symmetric matrices A_i . Let P^\perp be the orthogonal space of P with respect to the dual pairing between $\mathbb{S}^2 V$ and $\mathbb{S}^2 V^*$, and define $X = \mathcal{X} \cap P^\perp$. Dually, we may construct also Y in \mathcal{Y} as the pull-back of the quintic symmetroid $\mathcal{H} \cap P$. The linear system P is called *regular* if i) it is base point free and ii) any line $l \subset \text{Sing } Q$ for some $Q \in P$ is not contained in a linear subsystem of dimension ≥ 2 . X is smooth if and only if P is regular [HoTa1, Prop.2.1]. It has been shown that Y is also smooth for P is regular [ibid. Prop.3.11]. We say that X and Y defined for such a choice of P are *orthogonal* to each other.

Proposition 2.4.1. X and Y constructed as above are Calabi-Yau threefolds with the following invariants:

- 1) $\deg(X) = 35$, $c_2.D = 50$, $h^{2,1}(X) = 26$, $h^{1,1}(X) = 1$,
where D is the restriction to X of a hyperplane section of \mathcal{X} , $\deg(X) := D^3$ and c_2 is the second Chern class of X .
- 2) $\deg(Y) = 10$, $c_2.M = 40$, $h^{2,1}(Y) = 26$, $h^{1,1}(Y) = 1$,

where M is the restriction of $M_{\mathcal{Y}}$ to Y , $\deg(Y) := M^3$, and c_2 is the second Chern class of Y .

Proof. The invariants of X are easy to be determined, see [HoTa1, Prop.2.1]. The invariants of Y are determined in [HoTa3, Prop.4.3.4] (see also [HoTa1, Prop.3.11 and 3.12]). \square

Hereafter we consider the Calabi-Yau threefolds X and Y which are orthogonal to each other.

Since X is smooth, X is disjoint from $\text{Sing } \mathcal{X}$, and hence we can consider X to be contained in $\tilde{\mathcal{X}}$. Moreover, X is mapped by $g: \tilde{\mathcal{X}} \rightarrow G(2, V)$ onto its image isomorphically, and hence we can also consider X to be contained in $G(2, V)$. By the existence of this embedding into $G(2, V)$, X is called a (*generalized*) *Reye congruence*. As a subvariety of $G(2, V)$, X is characterized as the subset of lines l in $\mathbb{P}(V)$ such that quadrics which contain l form a 2-dimensional linear system (net) in P ([HoTa3, Prop.3.5.2]).

Y will be called the (*3-dimensional*) *double quintic symmetroid orthogonal to X* . Since Y is smooth, Y is disjoint from $\text{Sing } \mathcal{Y} = G_{\mathcal{Y}}$.

3. A FAMILY OF CURVES ON Y PARAMETERIZED BY X

In this section, using the generically conic bundle $\pi_{\mathcal{X}}: \mathcal{Z} \rightarrow \mathcal{Y}$, we construct a family of curves on Y of degree 5 parameterized by X , and show that its general member is a smooth curve of genus 3 for a general P .

Later in Section 8 (see also Section 10.1), we will show that this family is flat and explains the BPS number of curves of genus 3 and degree 5 on Y . The ideal sheaf of this family of curves in $Y \times X$ will be related with the birational model $G(3, T(-1)^{\wedge 2})$ of \mathcal{Y} and will give the kernel of a Fourier-Mukai functor which shows that $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

3.1. Constructing the family of curves.

Recall our definition of basic morphisms;

$$\mathcal{H} \xleftarrow{\rho_{\mathcal{Y}}} \mathcal{Y} \xleftarrow{\pi_{\mathcal{X}}} \mathcal{Z} \xrightarrow{\rho_{\mathcal{X}}} G(3, V),$$

where $\rho_{\mathcal{Z}}: \mathcal{Z} \rightarrow G(3, V)$ is a \mathbb{P}^8 -bundle since the fiber over a point $[\Pi]$ consists of quadrics which contain the plane $\mathbb{P}(\Pi)$.

We consider X in $G(2, V)$, and denote by l_x the line in $\mathbb{P}(V)$ which corresponds to a point $x \in X$. We set $P_x = \{[Q] \in P \mid l_x \subset Q\}$, the linear subsystem of P consisting of quadrics which contain the line l_x . Then $\dim P_x = 2$ holds [HoTa3, Prop.3.5.2]).

Lemma 3.1.1. *For any $x \in X$, the plane P_x is not contained in the quintic symmetroid $H := \mathcal{H} \cap P$. Moreover, for a general regular P and a general $x \in X$, the curve $H \cap P_x$ is a plane quintic with only three nodes.*

Proof. If $P_x \subset H$, then it is a divisor on H and $\rho_{\mathcal{Y}}^{-1}(P_x) = aM$ with some integer a , where M is the generator of $\text{Pic}(Y)$ and satisfies $M^3 = 10$. We set $M_H :=$

$\mathcal{O}_P(1)|_H$. By pulling back the intersection relation $1 = M_H \cdot M_H \cdot P_x$ to Y , we have $2 = M \cdot M \cdot (aM) = 10a$, which is absurd.

Note that H is a quintic hypersurface while $P_x \simeq \mathbb{P}^2$ is a linear subspace of P . Therefore $H \cap P_x$ is a plane quintic curve in P_x . The final part follows from an explicit calculation of the plane curve $H \cap P_x$ by *Macaulay2*. We verify in the example below that, for a general P and a general x , the curve $H \cap P_x$ has three nodes as singularities. \square

Example. (Nodal quintic curve $H \cap P_x$) We fix a generic (regular) linear system of quadrics $P = |Q_1, Q_2, \dots, Q_5|$ giving the quadratic forms $q_i(\mathbf{z}) = {}^t \mathbf{z} A_i \mathbf{z}$ on $\mathbb{P}(V)$ by 5×5 symmetric matrices. Explicitly we give them by

$$A_\lambda := \sum_{i=1}^5 \lambda_i A_i = \begin{pmatrix} \lambda_1 & \lambda_4 & \lambda_3 & \lambda_5 & \lambda_2 \\ \lambda_4 & -\lambda_3 & \lambda_2 - \lambda_5 & \lambda_2 & \lambda_4 \\ \lambda_3 & \lambda_2 - \lambda_5 & \lambda_2 & \lambda_4 & \lambda_1 + 2\lambda_2 \\ \lambda_5 & \lambda_2 & \lambda_4 & \lambda_1 & \lambda_4 \\ \lambda_2 & \lambda_4 & \lambda_1 + 2\lambda_2 & \lambda_4 & \lambda_1 + \lambda_2 \end{pmatrix}.$$

We identify $[A_\lambda]$ with the corresponding point $[\lambda] = [\sum_i \lambda_i Q_i]$ in P . Then it easy to verify that $[\mathbf{z}] = [-1, 0, 0, 1, 2]$ and $[\mathbf{w}] = [-1, 2, 0, -1, 0]$ satisfies ${}^t \mathbf{z} A_\lambda \mathbf{w} = 0$ for any $[\lambda] \in P$, hence defines a point x in X and also the corresponding line $l_x = \langle \mathbf{z}, \mathbf{w} \rangle$. The plane P_x is determined by the linear equations ${}^t \mathbf{z} A_\lambda \mathbf{z} = {}^t \mathbf{w} A_\lambda \mathbf{w} = 0$ as $P_x = \{3\lambda_1 + 2\lambda_4 - \lambda_5 = 2\lambda_1 - \lambda_2 - \lambda_3 = 0\} \subset P$. Then the curve $H \cap P_x$ is given by the quintic equation representing $P_x \cap \{\det A_\lambda = 0\}$. By calculating the Jacobian, it is straightforward to see that $H \cap P_x$ has three singularities at $[\lambda] = [1, \frac{2}{9}(2\alpha^2 + 3\alpha + 1), -\frac{2}{9}(2\alpha^2 + 3\alpha - 8), \alpha, 3 + 2\alpha]$ for each root α of the cubic $4x^3 - x^2 - 13x - 26 = 0$ which is nondegenerate. By writing the local equation of the curve, we verify that all these singularities are nodal.

The symmetroid $H = \mathcal{H} \cap P$ is written by $\{\det A_\lambda = 0\} \subset P$. By using *Macaulay2*, we verify that $\text{Sing } H$ is a smooth curve of genus 26 and degree 20 as noted in [HoTa1]. We also verify that $\text{Sing } H \cap (H \cap P_x) = \emptyset$.

Finally, consider a set $\{[\lambda] \in H \cap P_x \mid A_\lambda(a\mathbf{z} + b\mathbf{w}) = \mathbf{0}, \exists[a\mathbf{z} + b\mathbf{w}] \in l_x\}$, which represents quadrics which contain l_x with a point on l_x passing through their vertices. Note that by the regularity condition ii), there is no quadric that contains l_x in its vertex. We verify that the three nodes on $H \cap P_x$ exactly correspond to this set. \square

By this example, we see that the normalization of $H \cap P_x$ is a smooth curve of genus three for a general P and a general $x \in X$. We show that the normalization exists as curves on Z and Y .

To state the result precisely, we begin with a preliminary construction. We define

$$G_x := \{[\Pi] \in \mathbf{G}(3, V) \mid l_x \subset \mathbb{P}(\Pi)\}$$

and also

$$\mathcal{Z}_x := \{([\Pi], [Q]) \mid l_x \subset \mathbb{P}(\Pi) \subset Q\} = \rho_{\mathcal{Z}}^{-1}(G_x) \subset \mathcal{Z}.$$

G_x is a plane in $\mathbf{G}(3, V)$ and \mathcal{Z}_x is a \mathbb{P}^8 -bundle over G_x under the natural projection $\mathcal{Z}_x \rightarrow G_x$. Set

$$\gamma_x := \mathcal{Z}_x \cap \pi_{\mathcal{Z}}^{-1}(Y) = \{([\Pi], [Q]) \mid l_x \subset \mathbb{P}(\Pi) \subset Q, [Q] \in P\}$$

and denote by C_x its image on Y . We show

Proposition 3.1.2. *For smooth Calabi-Yau threefolds X and Y which are orthogonal to each other, $\{\gamma_x\}_{x \in X}$ is a family of curves of arithmetic genus 3 and of degree*

5 with respect to $M_{\mathcal{X}}$, and its images $\{C_x\}_{x \in X}$ on Y is a family of curves of degree 5 with respect to M .

Moreover, if X and Y are general, then a general member C_x is a smooth curve of genus 3.

Proof. Consider the projections $\overline{\gamma}_x := \rho_{\mathcal{Y}} \circ \pi_{\mathcal{X}}(\gamma_x)$ and $\mathcal{H}_x := \rho_{\mathcal{Y}} \circ \pi_{\mathcal{X}}(\mathcal{Z}_x)$. We define $\mathbb{P}_x := \{[Q] \in \mathbb{P}(\mathbb{S}^2 V^*) \mid l_x \subset Q\}$. If we write $x = w_{\mathbf{x}\mathbf{y}} \in X$ with $\mathbf{x}, \mathbf{y} \in \mathbb{P}(V)$ as a point of $\mathbb{S}^2 \mathbb{P}(V)$, then $\mathbb{P}_x = \{[Q] \in \mathbb{P}(\mathbb{S}^2 V^*) \mid {}^t \mathbf{x} A_Q \mathbf{x} = {}^t \mathbf{y} A_Q \mathbf{y} = {}^t \mathbf{x} A_Q \mathbf{y} = 0\}$, where A_Q is a 5×5 symmetric matrix defining the quadric Q . In particular, \mathbb{P}_x is isomorphic to \mathbb{P}^{11} . Then we have $\mathcal{H}_x = \{[Q] \mid l_x \subset \exists \mathbb{P}(\Pi) \subset Q\}$ and $\overline{\gamma}_x = \mathcal{H}_x \cap P = \mathcal{H}_x \cap \mathbb{P}_x$. If a singular quadric $[Q] \in \mathcal{H}$ contains a line l , then there always exists at least one plane $\mathbb{P}(\Pi)$ such that $l \subset \mathbb{P}(\Pi) \subset Q$. Hence we have $\mathcal{H}_x = \mathcal{H} \cap \mathbb{P}_x$ and $\overline{\gamma}_x = \mathcal{H}_x \cap P_x = \mathcal{H} \cap P_x \subset \mathcal{H} \cap P$. Since $H = \mathcal{H} \cap P$, we have $\overline{\gamma}_x = H \cap P_x$, which is a plane quintic curve by Lemma 3.1.1.

Now we note that if a line l is contained in a singular quadric Q but $l \not\subset \text{Sing } Q$, then there are at most two planes satisfying $l \subset \mathbb{P}(\Pi) \subset Q$. For the lines l_x of $x \in X$, we have $\dim P_x = 2$ as we see above. By the condition ii) of the regularity of P (see the beginning of Section 2.4), there is no quadric $[Q] \in P$ which contains the line l_x in $\text{Sing } Q$. Therefore $\gamma_x \rightarrow \overline{\gamma}_x$ is finite of degree at most two. In particular, γ_x is a curve.

By [HoTa3, Prop.3.5.2], P_x is a plane in \mathbb{P}_x and hence P_x is of codimension 9 in \mathbb{P}_x . Therefore, since $\overline{\gamma}_x = \mathcal{H}_x \cap P_x \subset \mathcal{H}_x \cap \mathbb{P}_x = \mathcal{H}_x$, we see that $\overline{\gamma}_x$ is also a complete intersection in $\mathcal{H}_x = \rho_{\mathcal{Y}} \circ \pi_{\mathcal{X}}(\mathcal{Z}_x)$ by 9 hyperplane sections. Corresponding to this, we also see that γ_x is a complete intersection of 9 elements of $|M_{\mathcal{X}}| := |M_{\mathcal{X}}|_{\mathcal{Z}_x}|$ in \mathcal{Z}_x since γ_x is the pull-back of $\overline{\gamma}_x$.

By this fact, we can compute the degree and the arithmetic genus of γ_x . The degree of γ_x with respect to $M_{\mathcal{X}}$ is evaluated by using $\mathcal{Z}_x = \rho_{\mathcal{X}}^{-1}(G_x)$ and the Segre class of the projective bundle $\mathcal{Z} = \mathbb{P}(\mathcal{E}^*) \rightarrow G(3, V)$ as

$$M_{\mathcal{X}} \cdot (\mathcal{Z}_x \cdot M_{\mathcal{X}}^9) = M_{\mathcal{X}}^{10} \cdot \mathcal{Z}_x = s_2(\mathcal{E}|_{G_x}) = (c_1(\mathcal{E})^2 - c_2(\mathcal{E}))G_x,$$

which is equal to $(c_1(\mathcal{E})^2 - c_2(\mathcal{E}))G_x = (10c_1(\mathcal{O}_{G(3,V)}(1))^2 - 5c_2(\mathcal{W}))G_x$ by Lemma 2.3.1. Since G_x is a plane, we have $c_1(\mathcal{O}_{G(3,V)}(1))^2 G_x = 1$. We note that, by definition, $c_2(\mathcal{W}) = \sigma_2$ which represents the 4-cycle $\{[\Pi] \mid t \in \mathbb{P}(\Pi)\} \subset G(3, V)$, parameterizing 2-planes containing a fixed point t of \mathbb{P}^4 . Therefore choosing such a point $t \in \mathbb{P}^4$ so that $t \notin l_x$, we see $c_2(\mathcal{W})G_x = 1$. Hence we have $M_{\mathcal{X}}^{10} \cdot \mathcal{Z}_x = M_{\mathcal{X}} \cdot \gamma_x = 5$. Since $\deg \gamma_x = \deg \overline{\gamma}_x = 5$, we see that $\gamma_x \rightarrow \overline{\gamma}_x$ is birational. The canonical divisor of γ_x is the restriction of $K_{\mathcal{Z}_x} + 9M_{\mathcal{X}}$. From the relative Euler sequence of the projective bundle $\mathcal{Z}_x = \mathbb{P}(\mathcal{E}^*|_{G_x})$ over $G_x \simeq \mathbb{P}^2$ and $c_1(\mathcal{E}) = c_1(\mathcal{O}_{G(3,V)}(4))$, we have $K_{\mathcal{Z}_x} = (-9M_{\mathcal{X}} + N_{\mathcal{X}})|_{\mathcal{Z}_x}$, where $N_{\mathcal{X}} := \rho_{\mathcal{X}}^* \mathcal{O}_{G(3,V)}(1)$. Thus $K_{\gamma_x} = N_{\mathcal{X}}|_{\gamma_x}$. Using the Segre class again, we evaluate

$$\rho_{\mathcal{X}*}(M_{\mathcal{X}}^9 \cdot \mathcal{Z}_x) = s_1(\mathcal{E}|_{G_x}) = c_1(\mathcal{E}|_{G_x}) = 4N_{\mathcal{X}}|_{G_x},$$

and obtain $\deg K_{\gamma_x} = N_{\mathcal{X}} M_{\mathcal{X}}^9 \cdot \mathcal{Z}_x = 4(N_{\mathcal{X}})^2|_{G_x} = 4$. Therefore the arithmetic genus of γ_x is 3.

Now we consider the image C_x on Y of γ_x . Note that a point $([\Pi], [Q])$ of γ_x satisfying $l_x \subset \mathbb{P}(\Pi) \subset Q$ ($[Q] \in P$) is mapped to a point $([Q], q)$ in Y , where q represents a connected family of planes contained in Q . Then $\gamma_x \rightarrow C_x$ is injective since once we fix a quadric Q of rank 3 or 4 and a connected family q therein, there exists at most one point $([\Pi], [Q])$ which satisfies $[\Pi] \in q$ and $l_x \subset \mathbb{P}(\Pi)$. In particular, the degree of C_x is 5 with respect to M .

Now we assume that X and Y are general. By Lemma 3.1.1, the geometric genus of $\overline{\gamma}_x$ is three for a general x . Since the arithmetic genus of γ_x is three, γ_x is the normalization of $\overline{\gamma}_x$. Since $\overline{\gamma}_x$ has only nodes as its singularities and $\gamma_x \rightarrow C_x$ is injective, we conclude $C_x \simeq \gamma_x$ and C_x is a smooth curve of genus 3 for general $x \in X$. \square

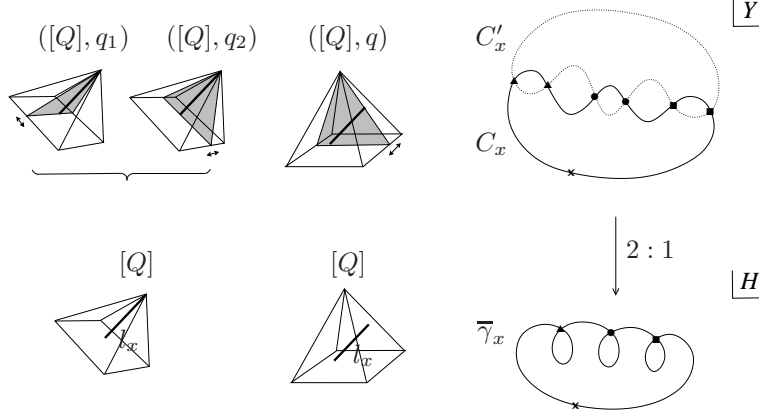


Fig.1. The curve C_x and its “shadow” C'_x . The line l_x and quadrics which contain l_x . If l_x passes through the vertex of Q , then two points $([Q], q_1), ([Q], q_2)$ map to $[Q]$. Otherwise, $([Q], q) \in Y$ is uniquely determined by $[Q]$.

Assume that X and Y are general. For a general point x in X , we can verify in the above example that the plane quintic curve $\overline{\gamma}_x$ has only three nodes and does not intersect with the singular locus $\text{Sing } H$. Since $Y \rightarrow H$ is a double cover branched along $\text{Sing } H$ and also the smooth curve C_x of degree 5 covers $\overline{\gamma}_x$, the inverse image $q^{-1}(\overline{\gamma}_x)$ is the union of C_x and another curve C'_x , which we called “shadow” curve of C_x in Introduction (see Fig.1). As shown in Fig.1, we note that the shadow curve is also a smooth curve of genus 3 and degree 5 for a general x and intersects at 6 points with C_x . These 6 points are inverse images of three nodal points on $\overline{\gamma}_x$.

3.2. The Brauer group of Y .

As an interesting corollary to the existence of the curves γ_x on Z , we show that Y has non-trivial Brauer group. Let $N_Z := N_{\mathcal{Z}}|_Z$ for $N_{\mathcal{Z}} = \rho_{\mathcal{Z}}^* \mathcal{O}_{G(3,V)}(1)$.

Proposition 3.2.1. *The \mathbb{P}^1 -fibration $Z \rightarrow Y$ is not associated to a locally free sheaf of rank two on Y . In particular, the Brauer group of Y contains a non-trivial 2-torsion element.*

Proof. Assume by contradiction that $Z = \mathbb{P}(\mathcal{A})$ for some locally free sheaf \mathcal{A} of rank two on Y . Since a fiber of $Z \rightarrow Y$ is of degree two with respect to N_Z , we can write $N_Z \equiv 2H_{\mathbb{P}(\mathcal{A})} + aM_Y$, where a is an integer (here we use $\rho(Y) = 1$). Since $N_Z \cdot \gamma_x = 4$ and $M_Y \cdot \gamma_x = 5$ by the proof of Proposition 3.1.2, we have

$4 = 2H_{\mathbb{P}(\mathcal{A})} \cdot \gamma'_x + 5a$. Thus a is even and $\frac{1}{2}N_Z$ is numerically equivalent to the Cartier divisor $H_{\mathbb{P}(\mathcal{A})} + \frac{1}{2}aM_Y$. Note that

$$(N_Z)^4 = N_{\mathcal{Z}}^4 M_{\mathcal{Z}}^{10} = s_2(\mathcal{E})c_1(\mathcal{O}_{G(3,V)}(1))^4 = (c_1(\mathcal{E})^2 - c_2(\mathcal{E}))c_1(\mathcal{O}_{G(3,V)}(1))^4.$$

By Lemma 2.3.1, we have $(N_Z)^4 = 40$. Then $(\frac{1}{2}N_Z)^4$ is not an integer, a contradiction. \square

We present a further discussion on the Brauer groups of X and Y in Subsection 10.2.

4. BIRATIONAL GEOMETRY OF \mathcal{Y}

Let $\mathcal{Y}_3 := G(3, T(-1)^{\wedge 2})$, which is a $G(3, 6)$ -bundle over $\mathbb{P}(V)$. The fiber of $\mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ over a point $[V_1] \in \mathbb{P}(V)$ parameterizes planes in $\mathbb{P}(\wedge^2(V/V_1))$. In [HoTa3], we have considered the two-ray game starting from the Mori fiber space $\mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ and have constructed the Sarkisov link. As a final step of the link, we have obtained a nice desingularization $\widetilde{\mathcal{Y}}$ of \mathcal{Y} , which appears in the following diagram:

$$(4.1) \quad \begin{array}{ccccccc} & & \mathcal{Z}_3 & & \mathcal{Y}_2 & & \mathcal{Y}_0 & & \mathcal{Z} \\ & \swarrow \rho_{\mathcal{Z}_3} & \downarrow \pi_{\mathcal{Z}_3} & \swarrow \rho_{\mathcal{Y}_2} & \downarrow q_{\mathcal{Y}_2} & \swarrow p_{\mathcal{Y}_2} & \downarrow & \swarrow & \downarrow \pi_{\mathcal{Z}} \\ & G(2, T(-1)) & \mathcal{Y}_3 & \cdots \rightarrow & \mathcal{U} & \xrightarrow{\tilde{\rho}_{\mathcal{Y}_2}} & \widetilde{\mathcal{Y}} & \xrightarrow{\rho_{\widetilde{\mathcal{Y}}}} & \mathcal{Y} \\ \swarrow \rho_G & \downarrow \pi_G & \downarrow \pi_{\mathcal{Y}_3} & \swarrow \pi_{\mathcal{U}} & \searrow p & & & & \downarrow q \\ G(3, V) & \mathbb{P}(V) & & & & & & & \mathcal{H} \end{array}$$

In this section, we review the construction and its several byproducts.

4.1. The two-ray game.

To start the two-ray game, we describe the contraction morphism from \mathcal{Y}_3 to $\overline{\mathcal{Y}}$ in Fig.2. For this, we set

$$\mathcal{P} := \{(\mathbb{P}^2, [V_1]) \mid \mathbb{P}^2 \text{ is a plane in } G(2, V/V_1)\} \subset \mathcal{Y}_3,$$

which is the orthogonal Grassmann bundle and consists of two connected components;

$$\mathcal{P} = \mathcal{P}_\rho \sqcup \mathcal{P}_\sigma.$$

The fiber of $\mathcal{P}_\rho \rightarrow \mathbb{P}(V)$ over a point $[V_1] \in \mathbb{P}(V)$ parameterizes planes of the form $P_{V_2/V_1} = \{[\mathbb{C}^2] \mid V_2/V_1 \subset \mathbb{C}^2\}$ for a two-dimensional subspace $V_2 \subset V$ such that $V_1 \subset V_2$. By construction, the fiber is isomorphic to $\mathbb{P}(V/V_1)$. Similarly, the fiber of $\mathcal{P}_\sigma \rightarrow \mathbb{P}(V)$ over a point $[V_1] \in \mathbb{P}(V)$ parameterizes planes of the form $P_{V_4/V_1} = \{[\overline{V}_2] \mid V_4/V_1 \supset \overline{V}_2\}$ for a four-dimensional subspace $V_4 \subset V$ such that $V_1 \subset V_4$. We see that the fiber is isomorphic to $\mathbb{P}((V/V_1)^*)$. Therefore we have

$$\mathcal{P}_\rho = v_2(\mathbb{P}(T(-1))) \text{ and } \mathcal{P}_\sigma = v_2(\mathbb{P}(\Omega(1))),$$

where v_2 means the relative second Veronese embedding.

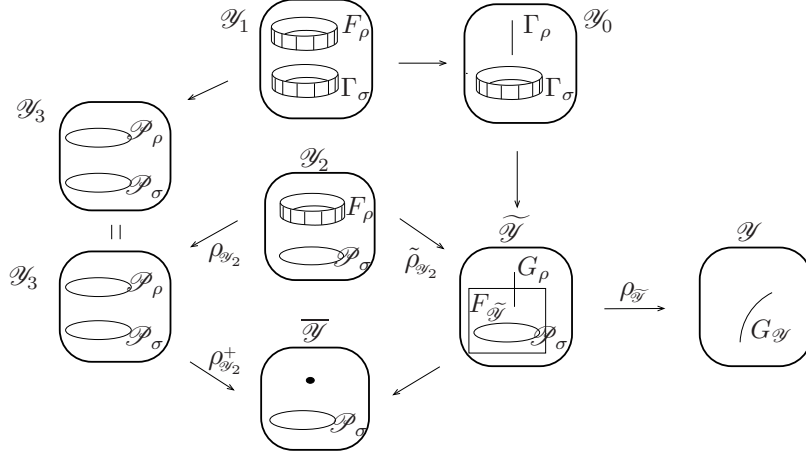


Fig.2. Birational geometries of \mathcal{Y} . The Sarkisov link is shown schematically in the square of the diagram.

Note that $\mathcal{P}_\rho \simeq \mathbb{P}(T(-1))$ can be identified with the flag variety $F(1, 2, V)$ and then there exists a unique morphism $\mathcal{P}_\rho \rightarrow G(2, V)$, which is a \mathbb{P}^1 -bundle. In [HoTa3, Sect.6.5], we have shown that \mathcal{P}_ρ is the exceptional locus of the $K_{\mathcal{Y}_3}$ -positive small contraction $\rho_{\mathcal{Y}_3}^+ : \mathcal{Y}_3 \rightarrow \widetilde{\mathcal{Y}}$ and $\rho_{\mathcal{Y}_3}^+$ induces the above morphism $\mathcal{P}_\rho \rightarrow G(2, V)$.

In [ibid. Sect.6.6], we have constructed the (anti-)flip $\mathcal{Y}_3 \dashrightarrow \widetilde{\mathcal{Y}}$ for $\rho_{\mathcal{Y}_3}^+$ and shown that $\widetilde{\mathcal{Y}}$ is smooth. Indeed, let $\mathcal{Y}_2 \rightarrow \mathcal{Y}_3$ be the blow-up of \mathcal{Y}_3 along \mathcal{P}_ρ and F_ρ the exceptional divisor. Then we have obtained the following commutative diagram:

$$(4.2) \quad \begin{array}{ccc} F_\rho & \xrightarrow{\quad} & G_\rho \\ \downarrow & & \downarrow \\ \mathcal{P}_\rho & \xrightarrow{\quad} & G(2, V), \end{array}$$

where $G_\rho \simeq \mathbb{P}(S^2\mathcal{G}^*)$ (see (2.1) for the definition of \mathcal{G}). We have shown that there exists a unique divisorial contraction $\tilde{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$ contracting F_ρ to G_ρ , and $\widetilde{\mathcal{Y}}$ is smooth. Actually, $\tilde{\rho}_{\mathcal{Y}_2}$ is the blow-up of $\widetilde{\mathcal{Y}}$ along G_ρ .

As a byproduct of the construction of \mathcal{Y}_2 , we have shown that there exists a morphism $\mathcal{Y}_2 \rightarrow \mathcal{U}$ ([ibid. Sect.6.4]). Indeed, noting the decomposition

$$\wedge^3(T(-1)^{\wedge 2}) = \left(S^2(T(-1)) \otimes \mathcal{O}(1) \right) \oplus \left(S^2(T(-1))^* \otimes \mathcal{O}(2) \right)$$

([ibid. Prop.6.3.1]), we have obtained a rational map $\mathcal{Y}_3 \dashrightarrow \mathcal{U} = \mathbb{P}(S^2\Omega(1))$, which is nothing but the projection from \mathcal{P}_ρ . Then the morphism $\mathcal{Y}_2 \rightarrow \mathcal{U}$ is obtained as a resolution of indeterminacy of $\mathcal{Y}_3 \dashrightarrow \mathcal{U}$. Then we have obtained a birational morphism $\mathcal{Y}_2 \rightarrow \mathcal{Y}$ from the Stein factorization of the composite $\mathcal{Y}_2 \rightarrow \mathcal{U} \rightarrow \mathcal{H}$ ([ibid. Prop.6.4.3]). Actually we have also proved that $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$ factors $\mathcal{Y}_2 \rightarrow \mathcal{Y}$. Thus it induces the desired morphism $\rho_{\widetilde{\mathcal{Y}}} : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$.

As a summary, we have obtained the following Sarkisov link:

$$\begin{array}{ccccc}
 \mathcal{P}_\rho \subset \mathcal{Y}_3 & \xrightarrow{\text{(anti-)flip}} & \widetilde{\mathcal{Y}} & \supset G_\rho \\
 \downarrow \text{G(3,6)-bundle} & \searrow & \swarrow & \downarrow \text{div. cont.} \\
 \mathbb{P}(V) & & \mathcal{Y} &
 \end{array}$$

In [ibid. Subsect.6.8 and Sect.7], we have described $\rho_{\widetilde{\mathcal{Y}}}: \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ in detail. We do not need such detail descriptions in this paper since they are needed only for computations of cohomology groups of certain locally free sheaves on $\widetilde{\mathcal{Y}}$ in [ibid. Sect.8]. We only mention that $\rho_{\widetilde{\mathcal{Y}}}$ is well described by relating $\widetilde{\mathcal{Y}}$ with the Hilbert scheme of conics on $\text{G}(3, V)$ (see the beginning of Section 5). Roughly speaking, $\rho_{\widetilde{\mathcal{Y}}}$ is the contraction of the prime divisor $F_{\widetilde{\mathcal{Y}}}$ parameterizing reducible conics on $\text{G}(3, V)$ to the locus $G_{\mathcal{Y}}$ ($\rho_{\widetilde{\mathcal{Y}}}$ is an isomorphism outside $F_{\widetilde{\mathcal{Y}}}$). The fiber of $\rho_{\widetilde{\mathcal{Y}}}$ over a point of $G_{\mathcal{Y}}$ corresponding to a quadric of rank 2 is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$.

Now we list up several byproducts of the construction of the Sarkisov link, which we need in the sequel. In what follows, we will use the following conventions without mentioning at each time:

- L_Σ : the pull back on a variety Σ of $\mathcal{O}(1)$ if there is a morphism $\Sigma \rightarrow \mathbb{P}(V)$.
- M_Σ : the pull back on a variety Σ of $\mathcal{O}_{\mathcal{H}}(1)$ if there is a morphism $\Sigma \rightarrow \mathcal{H}$.
- N_Σ : the pull back on a variety Σ of $\mathcal{O}_{\text{G}(3,V)}(1)$ if there is a morphism $\Sigma \rightarrow \text{G}(3, V)$.

- We have the following universal exact sequence on $\mathcal{Y}_3 = \text{G}(3, T(-1)^{\wedge 2})$:

$$(4.3) \quad 0 \rightarrow \mathcal{S}^* \rightarrow \pi_{\mathcal{Y}_3}^*(T(-1)^{\wedge 2}) \rightarrow \mathcal{Q} \rightarrow 0,$$

where \mathcal{S} is the dual of the relative universal subbundle of rank three and \mathcal{Q} is the relative universal quotient bundle of rank three.

Taking the determinant, we have

$$(4.4) \quad \det \mathcal{Q} = \det \mathcal{S} + 3L_{\mathcal{Y}_3} = \det\{\mathcal{S}(L_{\mathcal{Y}_3})\}.$$

We also have

$$(4.5) \quad K_{\mathcal{Y}_3} = -6 \det \mathcal{Q} + 4L_{\mathcal{Y}_3}.$$

- We list up some descriptions of $\mathcal{Q}|_{\mathcal{P}_\rho}$, which are important in calculations in the proof of Lemma 6.3.11.

$$(4.6) \quad \mathcal{Q}|_{\mathcal{P}_\rho} \simeq \mathcal{S}(L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho} \text{ ([ibid. Prop.6.3.2])}.$$

$$(4.7) \quad \det \mathcal{Q}|_{\mathcal{P}_\rho} = 2(H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho}) \text{ ([ibid. Prop.6.3.3])}.$$

- By the construction of $\mathcal{Y}_2 \rightarrow \mathcal{U}$, we obtain the following:

$$(4.8) \quad M_{\mathcal{Y}_2} = \rho_{\mathcal{Y}_2}^*(\det \mathcal{Q}) - L_{\mathcal{Y}_2} - F_\rho \text{ ([ibid. Prop.6.4.5])}.$$

4.2. The Grassmann bundle $G(2, T(-1))$.

Here we elaborate the following part of the diagram (4.1):

$$(4.9) \quad \begin{array}{ccc} & G(2, T(-1)) & \\ \swarrow \rho_G & & \searrow \pi_G \\ G(3, V) & & \mathbb{P}(V). \end{array}$$

Fix a point $[V_1] \in \mathbb{P}(V)$. Then the Euler sequence $0 \rightarrow \mathcal{O}(-1) \rightarrow V \otimes \mathcal{O} \rightarrow T(-1) \rightarrow 0$ over $[V_1]$ is represented by $0 \rightarrow V_1 \rightarrow V \rightarrow V/V_1 \rightarrow 0$. We denote the projection by $\pi_{V_1} : V \rightarrow V/V_1$. We consider the dual vector space V^* to V and identify the dual $(V/V_1)^*$ in V^* as $\{\varphi \in V^* \mid \varphi|_{V_1} = 0\}$. Then it is easy to deduce the following isomorphisms:

$$(4.10) \quad \begin{array}{ccc} ([V_1], [\overline{V}_2]) \in G(2, T(-1)) & \xrightarrow{\simeq} & G(2, \Omega(1)) \ni ([V_1], [(\overline{V}_2)^\perp]) \\ \downarrow \rho_G & & \downarrow \\ [\pi_{V_1}^{-1}(\overline{V}_2)] \in G(3, V) & \xrightarrow{\simeq} & G(2, V^*) \ni [\pi_{V_1}^{-1}(\overline{V}_2)^\perp], \end{array}$$

where \overline{V}_2 is a two dimensional subspace in V/V_1 .

Note that Grassmann bundles $G(2, T(-1))$ and $G(2, \Omega(1))$ are embedded into the projective bundles $\mathbb{P}(T(-1)^2)$ and $\mathbb{P}(\Omega(1)^2)$, respectively.

Lemma 4.2.1. $\mathbb{P}(\Omega(1)^{\wedge 2}) \simeq \mathbb{P}(T(-1)^{\wedge 2})$, and $H_{\mathbb{P}(\Omega(1)^{\wedge 2})} - H_{\mathbb{P}(T(-1)^{\wedge 2})} = L_{\mathbb{P}(T(-1)^{\wedge 2})}$ and $L_{\mathbb{P}(\Omega(1)^{\wedge 2})} = L_{\mathbb{P}(T(-1)^{\wedge 2})}$ hold.

Proof. By the natural isomorphism $T(-1)^{\wedge 2} \simeq \Omega(1)^{\wedge 2} \otimes \wedge^4 T(-1) \simeq \Omega(1)^{\wedge 2} \otimes \mathcal{O}(1)$, we have the assertions. The relation $L_{\mathbb{P}(T(-1)^{\wedge 2})} = L_{\mathbb{P}(\Omega(1)^{\wedge 2})}$ is clear by definition. \square

Since $\mathcal{O}_{G(2, V^*)}(1) = \mathcal{O}_{G(3, V)}(1)$ in (4.10), we see $H_{\mathbb{P}(\Omega(1)^{\wedge 2})}|_{G(2, T(-1))} = \rho_G^* \mathcal{O}_{G(3, V)}(1)$. Hence from the above lemma, we obtain

Proposition 4.2.2. Define $N_{G(2, T(-1))} := \rho_G^* \mathcal{O}_{G(3, V)}(1)$. Then

$$(4.11) \quad N_{G(2, T(-1))} = (H_{\mathbb{P}(T(-1)^{\wedge 2})} + L_{\mathbb{P}(T(-1)^{\wedge 2})})|_{G(2, T(-1))}.$$

Note that $G(2, T(-1))$ is a divisor in $\mathbb{P}(T(-1)^{\wedge 2})$ since its restrictions to fibers are quadric hypersurfaces. We may describe the linear equivalence class of $G(2, T(-1))$.

Proposition 4.2.3. $G(2, T(-1)) \in |2H_{\mathbb{P}(T(-1)^{\wedge 2})} + L_{\mathbb{P}(T(-1)^{\wedge 2})}|$.

Proof. See [HoTa3, Prop.6.1.1]. \square

Finally, we read from (4.10) that

Proposition 4.2.4. $\rho_G : G(2, T(-1)) \rightarrow G(3, V)$ is the universal family of planes in $\mathbb{P}(V)$ parameterized by $G(3, V)$.

Proof. The fiber of ρ_G over a point $[V_3]$ obviously consists of $([V_1], [\overline{V}_2])$ which fits into $0 \rightarrow V_1 \rightarrow V_3 \rightarrow \overline{V}_2 \rightarrow 0$. This is described by $\mathbb{P}(V_3)$. \square

5. GENERICALLY CONIC BUNDLES

As we mentioned in the final part of Subsection 2.3, the Hilbert scheme \mathcal{Y}_0 of conics on $G(3, V)$ is birational to \mathcal{Y} . By [IM, Remark in the end of §3.1] and the construction of $\widetilde{\mathcal{Y}}$, $\mathcal{Y}_0 \rightarrow \widetilde{\mathcal{Y}}$ is the blow-up of $\widetilde{\mathcal{Y}}$ along \mathcal{P}_σ , and $\widetilde{\mathcal{Y}} \setminus \mathcal{P}_\sigma$ parameterizes τ - and σ -conics on $G(3, V)$ while \mathcal{P}_σ parameterizes σ -planes, where, for notational simplicity, we write the transforms of $\mathcal{P}_\sigma \subset \mathcal{Y}_3$ on $\widetilde{\mathcal{Y}}$ by the same \mathcal{P}_σ .

By the definition of Hilbert scheme, we have the universal family of conics $\mathcal{Z}_0 \rightarrow \mathcal{Y}_0$ abstractly. In this section, we construct birational models of this conic bundle over \mathcal{Y}_3 and \mathcal{Y}_2 explicitly, which are needed to construct and describe the closed subscheme Δ in $\mathcal{Y} \times \mathcal{X}$. As a byproduct of the construction, we also construct the universal family $\mathcal{Z}_0 \rightarrow \mathcal{Y}_0$ in an explicit way.

5.1. Conics on $G(3, V)$.

Here we review some basic properties of conics on $G(3, V)$.

We first summarize the classification of conics on $G(3, V)$ according to the type of planes contained in $G(3, V)$:

- i) (ρ -plane) planes which are written by

$$P_{V_2} := \{[\Pi] \in G(3, V) \mid V_2 \subset \Pi\} \cong \mathbb{P}^2$$

for some V_2 , or

- ii) (σ -plane) planes which are written by

$$P_{V_1 V_4} := \{[\Pi] \in G(3, V) \mid V_1 \subset \Pi \subset V_4\} \cong \mathbb{P}^2$$

for some V_1 and V_4 with $V_1 \subset V_4$.

For a conic q on $G(3, V) \subset \mathbb{P}(\wedge^3 V)$, there exists a unique plane $\mathbb{P}_q^2 \subset \mathbb{P}(\wedge^3 V)$ such that $q \subset \mathbb{P}_q^2$. Then, there are two possibilities: $\mathbb{P}_q^2 \subset G(3, V)$ or $\mathbb{P}_q^2 \not\subset G(3, V)$.

When $\mathbb{P}_q^2 \subset G(3, V)$ we call q a ρ -conic if $\mathbb{P}_q^2 = P_{V_2}$ for some V_2 , and σ -conic if $\mathbb{P}_q^2 = P_{V_1 V_4}$ for some V_1 and V_4 . When $\mathbb{P}_q^2 \not\subset G(3, V)$, we have $q = \mathbb{P}_q^2 \cap G(3, V)$, and we call such a conic a τ -conic.

The Hilbert scheme \mathcal{Y}_0 of conics on $G(3, V)$ may be described as follows ([IM, Remark in the end of §3.1]): let $\mathcal{Y}_1 \rightarrow \mathcal{Y}_3$ be the blow-up along $\mathcal{P}_\rho \cup \mathcal{P}_\sigma$. Then the exceptional divisor over \mathcal{P}_ρ is isomorphic to F_ρ as in (4.2). It can be contracted in the other direction and the target of the contraction is nothing but \mathcal{Y}_0 . Then the locus $\Gamma_\rho \subset \mathcal{Y}_0$ of ρ -conics is the image of the exceptional divisor over \mathcal{P}_ρ , which is isomorphic to $G_\rho \simeq \mathbb{P}(S^2 \mathcal{G}^*)$, and the locus $\Gamma_\sigma \subset \mathcal{Y}_0$ of σ -conics is isomorphic to the exceptional divisor of $\mathcal{Y}_1 \rightarrow \mathcal{Y}_3$ over \mathcal{P}_σ . \mathcal{Y}_1 is also the blow-up of \mathcal{Y}_0 along Γ_ρ .

It has been shown in [HoTa3, Sect.5.3] that a smooth conic $q \subset G(3, V)$ is a τ -conic (resp. a ρ -conic) if and only if q is the fiber of $\pi_{\mathcal{X}}: \mathcal{Z} \rightarrow \mathcal{Y}$ over a rank 4 point (resp. a rank 3 points). Moreover, we have identified $\pi_{\mathcal{X}}: \mathcal{Z} \rightarrow \mathcal{Y}$ outside $G_{\mathcal{Y}}$ with the universal family of conics $\pi_{\mathcal{Z}_0}: \mathcal{Z}_0 \rightarrow \mathcal{Y}_0$.

5.2. A generically conic bundle $\pi_{\mathcal{Z}_3}: \mathcal{Z}_3 \rightarrow \mathcal{Y}_3$.

Set $\mathcal{Z}_3^u = \mathbb{P}(\mathcal{S}^*)$. The natural map $\pi_{\mathcal{Z}_3^u}: \mathcal{Z}_3^u \rightarrow \mathbf{G}(3, T(-1)^{\wedge 2})$ is the universal family of planes in the fibers of the \mathbb{P}^5 -bundle $\mathbb{P}(T(-1)^{\wedge 2}) \rightarrow \mathbb{P}(V)$.

$$(5.1) \quad \begin{array}{ccc} & \mathcal{Z}_3^u & \\ \rho_{\mathcal{Z}_3^u} \swarrow & & \searrow \pi_{\mathcal{Z}_3^u} \\ \mathbb{P}(T(-1)^{\wedge 2}) & & \mathcal{Y}_3 = \mathbf{G}(3, T(-1)^{\wedge 2}) \\ \pi_{\mathbb{P}} \searrow & & \swarrow \pi_{\mathcal{Y}_3} \\ & \mathbb{P}(V) & \end{array}$$

Restricting the diagram (5.1) to $\mathbf{G}(2, T(-1)) \subset \mathbb{P}(T(-1)^{\wedge 2})$ and setting $\mathcal{Z}_3 := \mathbf{G}(2, T(-1)) \times_{\mathbb{P}(T(-1)^{\wedge 2})} \mathcal{Z}_3^u$, we obtain

$$(5.2) \quad \begin{array}{ccc} & \mathcal{Z}_3 & \\ \rho_{\mathcal{Z}_3} \swarrow & & \searrow \pi_{\mathcal{Z}_3} \\ \mathbf{G}(2, T(-1)) & & \mathcal{Y}_3 = \mathbf{G}(3, T(-1)^{\wedge 2}) \\ \pi_G \searrow & & \swarrow \pi_{\mathcal{Y}_3} \\ & \mathbb{P}(V) & \end{array}$$

Note that $\mathbf{G}(2, T(-1))$ is contained in $\mathbf{G}(3, V) \times \mathbb{P}(V)$ since $\mathbf{G}(2, T(-1)) \rightarrow \mathbf{G}(3, V)$ is the universal family of planes on $\mathbb{P}(V)$ (Proposition 4.2.4). Therefore, since $\mathcal{Z}_3 \subset \mathbf{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathcal{Y}_3$, \mathcal{Z}_3 is contained in $(\mathbf{G}(3, V) \times \mathbb{P}(V)) \times_{\mathbb{P}(V)} \mathcal{Y}_3 \simeq \mathbf{G}(3, V) \times \mathcal{Y}_3$.

Proposition 5.2.1. *$\pi_{\mathcal{Z}_3}: \mathcal{Z}_3 \rightarrow \mathcal{Y}_3$ is a generically conic bundle, where a fiber of $\pi_{\mathcal{Z}_3}$ is considered as a subvariety of $\mathbf{G}(3, V)$ by the embedding $\mathcal{Z}_3 \subset \mathbf{G}(3, V) \times \mathcal{Y}_3$.*

Moreover, $\pi_{\mathcal{Z}_3}: \mathcal{Z}_3 \rightarrow \mathcal{Y}_3$ coincides with the universal family of conics $\pi_{\mathcal{Z}_0}: \mathcal{Z}_0 \rightarrow \mathcal{Y}_0$ on $\mathcal{Y}_3 \setminus \mathcal{P}_\rho \cup \mathcal{P}_\sigma$, where we consider $\mathcal{Y}_3 \setminus (\mathcal{P}_\rho \cup \mathcal{P}_\sigma)$ as an open subset of \mathcal{Y}_0 .

Proof. We have only to describe fibers of $\pi_{\mathcal{Z}_3}$ since \mathcal{Y}_3 is isomorphic to the Hilbert scheme \mathcal{Y}_0 of conics outside $\mathcal{P}_\rho \cup \mathcal{P}_\sigma$ by [IM, Remark in the end of §3.1]. Take $[V_1] \in \mathbb{P}(V)$ and $[V_3] \in \mathbf{G}(3, \wedge^2 V/V_1)$. Then the fiber of $\pi_{\mathcal{Z}_3}$ over $[V_3]$ is isomorphic to $\mathbb{P}(V_3) \cap \mathbf{G}(2, V/V_1) \subset \mathbf{G}(3, V)$, which is a conic if $[V_3] \notin \mathcal{P}_\rho \cup \mathcal{P}_\sigma$, or is the plane $\mathbb{P}(V_3)$ if $[V_3] \in \mathcal{P}_\rho \cup \mathcal{P}_\sigma$ (note that $\mathbf{G}(2, V/V_1)$ is a quadric in $\mathbb{P}(\wedge^2(V/V_1))$). \square

Note that \mathcal{Z}_3^u is contained in $\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathcal{Y}_3$, which is a \mathbb{P}^5 -bundle over \mathcal{Y}_3 . For Lemma 6.3.9, we need a description of the ideal sheaf of \mathcal{Z}_3^u in $\mathbb{P}(T(-1)^2) \times_{\mathbb{P}(V)} \mathcal{Y}_3$ as a projective subbundle. For this we need the following standard lemma:

Lemma 5.2.2. *Let X be a variety, and $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ a short exact sequence of locally free \mathcal{O}_X -sheaves. Associated to the surjection $\mathcal{B} \rightarrow \mathcal{C}$, we may regard $\mathbb{P}(\mathcal{C}^*)$ as a subbundle of $\mathbb{P}(\mathcal{B}^*)$. As such, the subvariety $\mathbb{P}(\mathcal{C}^*)$ is the complete intersection with respect to a section of $\pi^* \mathcal{A}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)$, where π is the natural projection $\mathbb{P}(\mathcal{B}^*) \rightarrow X$.*

Proof. Let \mathcal{I} be the ideal sheaf of $\mathbb{P}(\mathcal{C}^*)$ in $\mathbb{P}(\mathcal{B}^*)$. We have a natural exact sequence $0 \rightarrow \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{C}^*)}(1) \rightarrow 0$. Pushing forward this on X , we have $0 \rightarrow \pi_*(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)) \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$. Therefore $\mathcal{A} \simeq \pi_*(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1))$. Moreover, by a natural map $\pi^* \pi_*(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)) \rightarrow \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)$, we obtain $\pi^* \mathcal{A} \rightarrow \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)$. Investigating this along fibers, we see this is surjective. \square

Applying this to \mathcal{Z}_3^u , we can derive the Koszul resolution of $\mathcal{O}_{\mathcal{Z}_3^u}$ as a $\mathcal{O}_{\mathbb{P}(T(-1)^2) \times_{\mathbb{P}(V)} \mathcal{Y}_3}$ -module:

Proposition 5.2.3. *The subvariety \mathcal{Z}_3^u in $\mathbb{P}(T(-1)^2) \times_{\mathbb{P}(V)} \mathcal{Y}_3$ is the complete intersection with respect to a section of $\mathcal{O}_{\mathbb{P}(T(-1)^2)}(1) \boxtimes \mathcal{Q}$, hence $\mathcal{O}_{\mathcal{Z}_3^u}$ has the following Koszul resolution:*

$$0 \rightarrow \wedge^3 \{ \mathcal{O}_{\mathbb{P}(T(-1)^2)}(-1) \boxtimes \mathcal{Q}^* \} \rightarrow \wedge^2 \{ \mathcal{O}_{\mathbb{P}(T(-1)^2)}(-1) \boxtimes \mathcal{Q}^* \} \rightarrow \mathcal{O}_{\mathbb{P}(T(-1)^2)}(-1) \boxtimes \mathcal{Q}^* \rightarrow \mathcal{O}_{\mathbb{P}(T(-1)^2) \times_{\mathbb{P}(V)} \mathcal{Y}_3} \rightarrow \mathcal{O}_{\mathcal{Z}_3^u} \rightarrow 0.$$

Proof. Since $\mathcal{Z}_3^u = \mathbb{P}(\mathcal{S}^*)$, the assertion follows by applying Lemma 5.2.2 to the dual of (4.3). \square

We calculate some basic divisors on \mathcal{Z}_3 .

Proposition 5.2.4. (1) $N_{\mathcal{Z}_3} = H_{\mathbb{P}(\mathcal{S}^*)}|_{\mathcal{Z}_3} + L_{\mathcal{Z}_3}$.

(2) The relative canonical divisor $K_{\mathcal{Z}_3/\mathcal{Y}_3} := K_{\mathcal{Z}_3} - \pi_{\mathcal{Z}_3}^* K_{\mathcal{Y}_3}$ is given by

$$K_{\mathcal{Z}_3/\mathcal{Y}_3} = \pi_{\mathcal{Z}_3}^* (\det \mathcal{Q} - L_{\mathcal{Y}_3}) - N_{\mathcal{Z}_3}.$$

Proof. (1) follows from (4.11) and Proposition 4.2.1 since $H_{\mathbb{P}(\mathcal{S}^*)}|_{\mathcal{Z}_3}$ is the pull-back of $H_{\mathbb{P}(T(-1)^{\wedge 2})}$ by (4.3).

Since $\mathcal{Z}_3^u = \mathbb{P}(\mathcal{S}^*)$, we have the Euler sequence $0 \rightarrow \mathcal{O}_{\mathcal{Z}_3^u}(-1) \rightarrow \pi_{\mathcal{Z}_3^u}^* \mathcal{S}^* \rightarrow T_{\mathcal{Z}_3^u/\mathcal{Y}_3}(-1) \rightarrow 0$ and then

$$K_{\mathcal{Z}_3^u/\mathcal{Y}_3} = -3H_{\mathbb{P}(\mathcal{S}^*)} + \pi_{\mathcal{Z}_3^u}^* \det \mathcal{S}.$$

Note the inclusion $i : G(2, T(-1)) \hookrightarrow \mathbb{P}(T(-1)^{\wedge 2})$ and the definition $\mathcal{Z}_3 = \mathcal{Z}_3^u \cap i(G(2, T(-1)))$. Then, by Proposition 4.2.3, we have $\mathcal{Z}_3 \in |2H_{\mathbb{P}(\mathcal{S}^*)} + L_{\mathcal{Z}_3^u}|$. Hence, by the adjunction formula and (4.5), we have

$$K_{\mathcal{Z}_3/\mathcal{Y}_3} = \{K_{\mathcal{Z}_3^u/\mathcal{Y}_3} + (2H_{\mathbb{P}(\mathcal{S}^*)} + L_{\mathcal{Z}_3^u})\}|_{\mathcal{Z}_3} = -H_{\mathbb{P}(\mathcal{S}^*)}|_{\mathcal{Z}_3} + L_{\mathcal{Z}_3} + \pi_{\mathcal{Z}_3}^* \det \mathcal{S}.$$

Finally, by (1) and (4.4), we obtain (2). \square

\mathcal{Z}_3 has the following nice description:

Proposition 5.2.5. $\rho_{\mathcal{Z}_3} : \mathcal{Z}_3 \rightarrow G(2, T(-1))$ is a $G(2, 5)$ -bundle. In particular, \mathcal{Z}_3 is smooth.

Proof. Take a point $([\overline{V}_2], [V_1]) \in G(2, T(-1))$, where $[\overline{V}_2] \in G(2, V/V_1)$ and $[V_1] \in \mathbb{P}(V)$. Then the fiber of $\rho_{\mathcal{Z}_3}$ over $([\overline{V}_2], [V_1])$ is isomorphic to $\{[V_3] \mid \wedge^2 \overline{V}_2 \subset V_3 \subset \wedge^2(V/V_1)\} \simeq G(2, \wedge^2(V/V_1)/\wedge^2 \overline{V}_2)$. \square

We set $\mathcal{Z}_\rho := \pi_{\mathcal{Z}_3}^{-1}(\mathcal{P}_\rho) \simeq \mathbb{P}(\mathcal{S}^*|_{\mathcal{P}_\rho})$ and $\mathcal{Z}_\sigma := \pi_{\mathcal{Z}_3}^{-1}(\mathcal{P}_\sigma) \simeq \mathbb{P}(\mathcal{S}^*|_{\mathcal{P}_\sigma})$. $\mathcal{Z}_\rho \rightarrow \mathcal{P}_\rho$ and $\mathcal{Z}_\sigma \rightarrow \mathcal{P}_\sigma$ are the family of planes parameterized by \mathcal{P}_ρ and \mathcal{P}_σ respectively. Now we describe the restriction of the diagram (5.2) over \mathcal{P}_ρ :

$$(5.3) \quad \begin{array}{ccccc} & & \mathcal{Z}_\rho & & \\ & \swarrow \rho_{\mathcal{Z}_\rho} & & \searrow \pi_{\mathcal{Z}_\rho} & \\ G(2, T(-1)) & & & & \mathcal{P}_\rho \\ & \searrow \pi_G & & \swarrow \pi_{\mathcal{P}_\rho} & \\ & & \mathbb{P}(V), & & \end{array}$$

where we set $\rho_{\mathcal{Z}_\rho} := \rho_{\mathcal{Z}_3}|_{\mathcal{Z}_\rho}$ and $\pi_{\mathcal{Z}_\rho} := \pi_{\mathcal{Z}_3}|_{\mathcal{Z}_\rho}$.

Proposition 5.2.6. $\rho_{\mathcal{Z}_\rho} : \mathcal{Z}_\rho \rightarrow \mathrm{G}(2, T(-1))$ is a \mathbb{P}^1 -bundle. If we consider $\rho_{\mathcal{Z}_\rho}$ as a sub-fibration of $\mathrm{G}(2, 5)$ -bundle $\rho_{\mathcal{Z}_3} : \mathcal{Z}_3 \rightarrow \mathrm{G}(2, T(-1))$, then a fiber of $\rho_{\mathcal{Z}_\rho}$ is a conic in $\mathrm{G}(2, 5)$. Similar assertions hold also for $\rho_{\mathcal{Z}_\sigma} := \rho_{\mathcal{Z}_3}|_{\mathcal{Z}_\sigma} : \mathcal{Z}_\sigma \rightarrow \mathrm{G}(2, T(-1))$.

Proof. Take a point $[V_3/V_1] \in \mathrm{G}(2, V/V_1) \subset \mathrm{G}(2, T(-1))$. We show the fiber γ of $\rho_{\mathcal{Z}_\rho}$ over $[V_3/V_1]$ is \mathbb{P}^1 . Note that γ can be considered as a subvarieties of a fiber of $\mathcal{P}_\rho \rightarrow \mathbb{P}(V)$ since $\mathcal{Z}_\rho \subset \mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathcal{P}_\rho$. Recall that the fiber of $\mathcal{P}_\rho \rightarrow \mathbb{P}(V)$ over a point $[V_1] \in \mathbb{P}(V)$ parameterizes planes in $\mathrm{G}(2, V/V_1)$ of the form $\mathrm{P}_{V_2/V_1} = \{[\overline{V}_2] \mid V_2/V_1 \subset \overline{V}_2\}$ with some 2-dimensional subspace $V_2 \subset V$. Then γ parameterizes planes of the form P_{V_2/V_1} which contain $[V_3/V_1] \in \mathrm{G}(2, V/V_1)$. Therefore $\gamma \simeq \{V_2 \mid V_1 \subset V_2 \subset V_3\} \simeq \mathbb{P}^1$. This can be identified with a line in $\mathbb{P}(V/V_1)$. Thus this is a conic in $\mathcal{P}_\rho \cap \mathrm{G}(3, \wedge^2 V/V_1) = v_2(\mathbb{P}(V/V_1))$.

The assertions for $\rho_{\mathcal{Z}_\sigma}$ can be proved similarly. □

Finally, for Lemma 6.3.11, we add a fact about the relative Euler sequence for the projective bundle $\pi_{\mathcal{Z}_\rho} : \mathbb{P}(\mathcal{S}^*|_{\mathcal{P}_\rho}) \rightarrow \mathcal{P}_\rho$;

$$(5.4) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{S}^*|_{\mathcal{P}_\rho})}(-1) \rightarrow \pi_{\mathcal{Z}_\rho}^*(\mathcal{S}^*|_{\mathcal{P}_\rho}) \rightarrow \mathcal{R}_{\mathcal{Z}_\rho} \rightarrow 0,$$

where we set $\mathcal{R}_{\mathcal{Z}_\rho} := T_{\mathbb{P}(\mathcal{S}^*|_{\mathcal{P}_\rho})/\mathcal{P}_\rho} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{S}^*|_{\mathcal{P}_\rho})}(-1)$.

Lemma 5.2.7. Let $\mathcal{W}_{\mathcal{Z}_\rho}$ be the pull-back of the universal quotient bundle \mathcal{W} on $\mathrm{G}(3, V)$ by the composite $\mathcal{Z}_\rho \rightarrow \mathrm{G}(2, T(-1)) \rightarrow \mathrm{G}(3, V)$. It holds that $\mathcal{W}_{\mathcal{Z}_\rho} \simeq \mathcal{R}_{\mathcal{Z}_\rho} \otimes \pi_{\mathcal{Z}_\rho}^* \mathcal{O}_{\mathbb{P}(T(-1))}(1)$.

Proof. Take a point $[P_{V_2/V_1}]$ of \mathcal{P}_ρ and then take a point $[\overline{V}_1]$ of the fiber $\mathbb{P}(V/V_2 \otimes V_2/V_1)$ of $\pi_{\mathcal{Z}_\rho}$ over $[P_{V_2/V_1}]$. The exact sequence (5.4) restricts at $[\overline{V}_1]$ to

$$0 \rightarrow \overline{V}_1 \rightarrow V/V_2 \otimes V_2/V_1 \rightarrow (V/V_2 \otimes V_2/V_1)/\overline{V}_1 \rightarrow 0.$$

On the other hand, let $[V_3/V_2]$ be the point of $\mathbb{P}(V/V_2)$ corresponding to $[\overline{V}_1]$ through the isomorphism $V/V_2 \simeq V/V_2 \otimes V_2/V_1$, namely, $V_3/V_2 \otimes V_2/V_1 = \overline{V}_1$. The fiber of $\mathcal{W}_{\mathcal{Z}_\rho}$ at $[\overline{V}_1]$ is nothing but $V/V_3 \simeq (V/V_2)/(V_3/V_2)$. Therefore we have $\mathcal{W}_{\mathcal{Z}_\rho} \simeq \mathcal{R}_{\mathcal{Z}_\rho} \otimes \pi_{\mathcal{Z}_\rho}^* \mathcal{O}_{\mathbb{P}(T(-1))}(1)$. □

5.3. Generically conic bundles $\pi_{\mathcal{Z}_2} : \mathcal{Z}_2 \rightarrow \mathcal{Y}_2$ and $\pi_{\mathcal{Z}_1} : \mathcal{Z}_1 \rightarrow \mathcal{Y}_1$.

In this subsection, we construct the generically conic bundle $\mathcal{Z}_2 \rightarrow \mathcal{Y}_2$ which is isomorphic to the universal family of conics on $\mathrm{G}(3, V)$ outside \mathcal{P}_σ .

Recall the diagram (5.1) and its restriction (5.2). In general, for a flat fibration $\pi : \Sigma \rightarrow \mathcal{Y}_3$, the morphism $\Sigma \times_{\mathcal{Y}_3} \mathcal{Y}_2 \rightarrow \Sigma$ is the blow-up along $\pi^{-1}(\mathcal{P}_\rho)$ and the exceptional divisor is $\Sigma \times_{\mathcal{Y}_3} F_\rho$ since $\mathcal{Y}_2 \rightarrow \mathcal{Y}_3$ is the blow-up along \mathcal{P}_ρ with the exceptional divisor F_ρ . We apply this fact to the \mathbb{P}^5 -bundle $\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathcal{Y}_2$ over \mathcal{Y}_2 , and also its two subbundles;

- (i) the $\mathrm{G}(2, 4)$ -bundle $G_2 := \mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathcal{Y}_2$ and
- (ii) the \mathbb{P}^2 -bundle $\mathcal{Z}_2^u := \mathcal{Z}_3^u \times_{\mathcal{Y}_3} \mathcal{Y}_2 = \mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*)$.

The exceptional divisor of $\mathcal{Z}_2^u \rightarrow \mathcal{Z}_3^u$ is $\mathcal{Z}_3^u \times_{\mathcal{Y}_3} F_\rho = \mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*|_{F_\rho})$. Note that $\mathcal{Z}_3 = (G(2, T(-1)) \times_{\mathbb{P}(V)} \mathcal{Y}_3) \cap \mathcal{Z}_3^u$ by definition of \mathcal{Z}_3 . Then the corresponding intersection

$$\mathcal{Z}_2^t := G_2 \cap \mathcal{Z}_2^u$$

is the total transform of \mathcal{Z}_3 since it contains the exceptional divisor of $\mathcal{Z}_2^u \rightarrow \mathcal{Z}_3^u$. \mathcal{Z}_2^t is reduced since \mathcal{Z}_3 is smooth by Proposition 5.2.5. This \mathcal{Z}_2^t also plays an important role for the proof of Proposition 6.3.10 (see Lemma 6.3.9).

Let \mathcal{Z}_2 be the strict transform of \mathcal{Z}_3 , which is nothing but the blow-up of \mathcal{Z}_3 along \mathcal{Z}_ρ by the universal property of blow-up. By Proposition 5.2.1, $\mathcal{Z}_3 \rightarrow \mathcal{Y}_3$ coincides with the universal family $\mathcal{Z}_0 \rightarrow \mathcal{Y}_0$ of conics on $G(3, V)$ outside \mathcal{P}_ρ and \mathcal{P}_σ , where we consider $\mathcal{Y}_3 \setminus (\mathcal{P}_\rho \cup \mathcal{P}_\sigma)$ as an open set of \mathcal{Y}_0 . Thus \mathcal{Z}_2 coincides with \mathcal{Z}_0 outside the inverse images of F_ρ and \mathcal{P}_σ , where we denote by \mathcal{P}_σ the transform in \mathcal{Y}_2 of \mathcal{P}_σ and we consider $\mathcal{Y}_2 \setminus (F_\rho \cup \mathcal{P}_\sigma)$ as an open set of \mathcal{Y}_0 . Recall that $\mathcal{Y}_1 \rightarrow \mathcal{Y}_0$ is the blow-up along the locus Γ_ρ of ρ -conics. Let \mathcal{Z}_1 be the pull-back of the universal family \mathcal{Z}_0 to \mathcal{Y}_1 . Now we consider $\mathcal{Y}_2 \setminus \mathcal{P}_\sigma$ as an open set of \mathcal{Y}_1 . Then, over $\mathcal{Y}_2 \setminus \mathcal{P}_\sigma$, \mathcal{Z}_2 coincides with \mathcal{Z}_1 since \mathcal{Z}_2 and \mathcal{Z}_1 are prime divisors in the \mathbb{P}^2 -bundle $\mathcal{Z}_2^u|_{\mathcal{Y}_2 \setminus \mathcal{P}_\sigma}$ and coincide with each other over $\mathcal{Y}_2 \setminus (\mathcal{P}_\sigma \cup F_\rho)$. In particular, the fiber of $\mathcal{Z}_2 \rightarrow \mathcal{Y}_2$ over a point of F_ρ can be regarded as a ρ -conic.

In a similar way, we can show \mathcal{Z}_1 is equal to the blow-up of \mathcal{Z}_3 along \mathcal{Z}_ρ and \mathcal{Z}_σ . Therefore \mathcal{Z}_0 can be constructed explicitly by contracting the inverse image of F_ρ in the other direction.

We calculate the relative canonical divisor $K_{\mathcal{Z}_2/\mathcal{Y}_2}$:

Proposition 5.3.1. *It holds $K_{\mathcal{Z}_2/\mathcal{Y}_2} = M_{\mathcal{Z}_2} - N_{\mathcal{Z}_2}$.*

Proof. Denote by E_ρ the exceptional divisor of $\rho_{\mathcal{Z}_2}: \mathcal{Z}_2 \rightarrow \mathcal{Z}_3$. Note that $\pi_{\mathcal{Z}_2}(E_\rho)$ is the exceptional divisor F_ρ of $\rho_{\mathcal{Y}_2}: \mathcal{Y}_2 \rightarrow \mathcal{Y}_3$ and $E_\rho = \pi_{\mathcal{Z}_2}^{-1}(F_\rho)$. Since the codimension of \mathcal{P}_ρ in \mathcal{Y}_3 is three and that of $\pi_{\mathcal{Z}_3}^{-1}(\mathcal{P}_\rho)$ in \mathcal{Z}_3 is two, we have

$$K_{\mathcal{Z}_2/\mathcal{Y}_2} = \rho_{\mathcal{Z}_2}^* K_{\mathcal{Z}_3/\mathcal{Y}_3} - E_\rho.$$

By Proposition 5.2.4 (2), it holds $K_{\mathcal{Z}_3/\mathcal{Y}_3} = \pi_{\mathcal{Z}_3}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3}) - N_{\mathcal{Z}_3}$. Therefore we have

$$K_{\mathcal{Z}_2/\mathcal{Y}_2} = \pi_{\mathcal{Z}_2}^*(\rho_{\mathcal{Y}_2}^* \det \mathcal{Q} - L_{\mathcal{Y}_2} - F_\rho) - N_{\mathcal{Z}_2}.$$

Now, by (4.8), we have the assertion. \square

Further, we may construct a generically conic bundle $\pi_{\widetilde{\mathcal{Z}}}: \widetilde{\mathcal{Z}} \rightarrow \widetilde{\mathcal{Y}}$ by replacing \mathcal{Z}_2 over F_ρ by \mathcal{Z}_0 . Then we obtain the following commutative diagram:

$$(5.5) \quad \begin{array}{ccc} \mathcal{Z}_2 & \longrightarrow & \widetilde{\mathcal{Z}} \\ \downarrow & & \downarrow \\ \mathcal{Y}_2 & \longrightarrow & \widetilde{\mathcal{Y}}. \end{array}$$

6. CONSTRUCTING THE IDEAL SHEAF \mathcal{I} OF A CLOSED SUBSCHEME Δ ON $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$

The aim of this section is to construct an ideal sheaf on $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$ with its explicit locally free resolution. This ideal sheaf entails the derived equivalence between X and Y in Section 9. Further we expect this would describe the homologically projective duality of suitable noncommutative resolutions of \mathcal{X} and \mathcal{Y} .

6.1. Statement of the result.

In [HoTa3, Subsect.6.7], we have introduced locally free sheaves \widetilde{S}_L , \widetilde{Q} and \widetilde{T} on $\widetilde{\mathcal{Y}}$. \widetilde{S}_L and \widetilde{Q} are locally free sheaves characterized by the properties that their pull-backs on \mathcal{Y}_2 coincides with $\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$ and $\rho_{\mathcal{Y}_2}^* \mathcal{Q}$ respectively. \widetilde{T} is a locally free sheaf such that the dual \mathcal{T}^* of its pull-back on \mathcal{Y}_2 fits into the exact sequence

$$(6.1) \quad 0 \rightarrow \mathcal{T}^* \rightarrow \pi_{\mathcal{Y}_2}^* \Omega(1) \rightarrow (\rho_{\mathcal{Y}_2}|_{F_p})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \rightarrow 0.$$

It has been shown that these sheaves define a Lefschetz collection in the derived category $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ [ibid. Thm.8.1.1]. In fact, these sheaves appear in the following resolution of an ideal sheaf \mathcal{I} on $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$.

Theorem 6.1.1. *There exists an $\mathrm{SL}(V)$ -invariant normal Cohen-Macaulay closed subvariety Δ in $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$ whose ideal sheaf \mathcal{I} has the following $\mathrm{SL}(V)$ -equivariant locally free resolution:*

$$(6.2) \quad \begin{aligned} 0 \rightarrow \widetilde{S}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}} \rightarrow \widetilde{T}^* \boxtimes g^* \mathcal{F} \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}} \boxtimes g^* \mathcal{S}^2 \mathcal{F} \oplus \widetilde{Q}^*(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(L_{\check{\mathcal{X}}}) \\ \rightarrow \mathcal{I} \otimes \{\mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\} \rightarrow 0. \end{aligned}$$

Our proof consists of several steps in the subsequent subsections and is completed in Subsection 6.5.

Before we proceed to the proof, we remark that the morphisms of the resolution (6.2) are determined uniquely by requiring $\mathrm{SL}(V)$ equivariance. For this, let us write the resolution (6.2) in terms of the ordered collections introduced in [HoTa3, Thm.3.4.4 and 8.1.1] with minor modification;

$$(6.3) \quad \begin{aligned} (\mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_{1a}, \mathcal{E}_{1b}) &= (\widetilde{S}_L^*, \widetilde{T}^*, \mathcal{O}_{\widetilde{\mathcal{Y}}}, \widetilde{Q}^*(M_{\widetilde{\mathcal{Y}}})), \\ (\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}'_{1a}, \mathcal{F}_{1b}) &= (\mathcal{O}_{\check{\mathcal{X}}}, g^* \mathcal{F}, g^* \mathcal{S}^2 \mathcal{F}, \mathcal{O}_{\check{\mathcal{X}}}(L_{\check{\mathcal{X}}})). \end{aligned}$$

Using these, the resolution takes a concise form;

$$0 \rightarrow \mathcal{E}_3 \boxtimes \mathcal{F}_3 \rightarrow \mathcal{E}_2 \boxtimes \mathcal{F}_2 \rightarrow \mathcal{E}_{1a} \boxtimes \mathcal{F}'_{1a} \oplus \mathcal{E}_{1b} \boxtimes \mathcal{F}_{1b} \rightarrow \mathcal{I} \otimes \{\mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\} \rightarrow 0.$$

Then the maps in the resolution are derived from

$$\begin{aligned} (a) \quad \mathcal{E}_3 \boxtimes \mathcal{F}_3 &\rightarrow \mathcal{E}_2 \boxtimes \mathcal{F}_2, \quad (b) \quad \mathcal{E}_2 \boxtimes \mathcal{F}_2 \rightarrow \mathcal{E}_{1a} \boxtimes \mathcal{F}'_{1a}, \quad (c) \quad \mathcal{E}_2 \boxtimes \mathcal{F}_2 \rightarrow \mathcal{E}_{1b} \boxtimes \mathcal{F}_{1b}, \\ (d) \quad \mathcal{E}_{1a} \boxtimes \mathcal{F}'_{1a} &\rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}}), \\ (e) \quad \mathcal{E}_{1b} \boxtimes \mathcal{F}_{1b} &\rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}}). \end{aligned}$$

We show that all these maps are constructed $\mathrm{SL}(V)$ -equivariantly by the following principle:

Let $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{B}_1, \mathcal{B}_2)$ be two pairs of locally free sheaves on a variety. By tensoring the evaluation maps $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2) \otimes \mathcal{A}_1 \rightarrow \mathcal{A}_2$ and $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2) \otimes \mathcal{B}_1 \rightarrow \mathcal{B}_2$, we obtain

$$(\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2) \otimes \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)) \otimes (\mathcal{A}_1 \otimes \mathcal{B}_1) \rightarrow \mathcal{A}_2 \otimes \mathcal{B}_2.$$

We suppose

$$(6.4) \quad \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2) \simeq \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)^*.$$

Then corresponding to the identity element in

$$\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2) \otimes \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2) \simeq \mathrm{Hom}(\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2), \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)),$$

we obtain the $\mathrm{SL}(V)$ -equivariant map

$$(6.5) \quad \mathcal{A}_1 \otimes \mathcal{B}_1 \rightarrow \mathcal{A}_2 \otimes \mathcal{B}_2.$$

We verify the condition (6.4) for the maps (a),(c) from the non-vanishing Hom 's among the the ordered collections $(\mathcal{E}_i)_{i \in I}$ and $(\mathcal{F}_i)_{i \in I}$ with $\mathcal{F}_{1a} = \mathcal{F}'_{1a}/\mathcal{O}_{\tilde{\mathcal{X}}}(-H_{\tilde{\mathcal{X}}} + 2L_{\tilde{\mathcal{X}}})$, which are displayed in the following quiver diagrams:

$$(6.6) \quad \begin{array}{ccc} & \begin{array}{c} \xrightarrow{\wedge^2 V} \circ \mathcal{E}_{1a} \\ \nearrow V \quad \searrow V \\ \circ \mathcal{E}_2 \\ \nwarrow V \quad \swarrow V \\ \xrightarrow{S^2 V} \circ \mathcal{E}_{1b} \end{array} & \begin{array}{c} \xrightarrow{S^2 V^*} \circ \mathcal{F}_{1a} \\ \nearrow V^* \quad \searrow V^* \\ \circ \mathcal{F}_2 \\ \nwarrow V^* \quad \swarrow V^* \\ \xrightarrow{\wedge^2 V^*} \circ \mathcal{F}_{1b} \end{array} \\ \mathrm{Hom}_{\mathcal{O}_{\tilde{\mathcal{Y}}}}(\mathcal{E}_i, \mathcal{E}_j) & & \mathrm{Hom}_{\mathcal{O}_{\tilde{\mathcal{X}}}}(\mathcal{F}_i, \mathcal{F}_j) \end{array}$$

For the case (b), the isomorphism $\mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_{1a}) \simeq V$ appears in the diagram (6.6). To compute $\mathrm{Hom}(\mathcal{F}_2, \mathcal{F}'_{1a})$, we use the exact sequence $0 \rightarrow \mathcal{O}_{\tilde{\mathcal{X}}}(-H_{\tilde{\mathcal{X}}} + 2L_{\tilde{\mathcal{X}}}) \rightarrow \mathcal{F}'_{1a} \rightarrow \mathcal{F}_{1a} \rightarrow 0$. The vanishing $H^\bullet(\tilde{\mathcal{X}}, g^* \mathcal{F}^* \otimes \mathcal{O}_{\tilde{\mathcal{X}}}(-H_{\tilde{\mathcal{X}}} + 2L_{\tilde{\mathcal{X}}})) = 0$ follows from the fact that $g : \tilde{\mathcal{X}} \rightarrow \mathrm{G}(2, V)$ is a \mathbb{P}^2 -bundle. Therefore we have $\mathrm{Hom}(\mathcal{F}_2, \mathcal{F}'_{1a}) \simeq \mathrm{Hom}(\mathcal{F}_2, \mathcal{F}_{1a}) \simeq V^*$, where the latter isomorphism appears in the diagram (6.6). For the cases (c) and (d), respectively, we use the following calculations:

$$\begin{aligned} \mathrm{Hom}(\mathrm{pr}_1^* \mathcal{E}_{1a}, \mathrm{pr}_1^* \mathcal{O}_{\tilde{\mathcal{Y}}}(M_{\tilde{\mathcal{Y}}})) &= H^0(\tilde{\mathcal{Y}}, \mathcal{O}_{\tilde{\mathcal{Y}}}(M_{\tilde{\mathcal{Y}}})) \simeq S^2 V, \\ \mathrm{Hom}(\mathrm{pr}_2^* \mathcal{F}'_{1a}, \mathrm{pr}_2^* \mathcal{O}_{\tilde{\mathcal{X}}}(2L_{\tilde{\mathcal{X}}})) &= H^0(\mathrm{G}(2, V), S^2 \mathcal{F}) \simeq S^2 V^*, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}(\mathrm{pr}_1^* \mathcal{E}_{1b}, \mathrm{pr}_1^* \mathcal{O}_{\tilde{\mathcal{Y}}}(M_{\tilde{\mathcal{Y}}})) &= H^0(\mathcal{Y}_3, \mathcal{Q}) = H^0(\mathbb{P}(V), T(-1)^2) \simeq \wedge^2 V^*, \\ \mathrm{Hom}(\mathrm{pr}_2^* \mathcal{F}_{1b}, \mathrm{pr}_2^* \mathcal{O}_{\tilde{\mathcal{X}}}(2L_{\tilde{\mathcal{X}}})) &= H^0(\mathrm{G}(2, V), \mathcal{O}_{\tilde{\mathcal{X}}}(L_{\tilde{\mathcal{X}}})) \simeq \wedge^2 V. \end{aligned}$$

Let $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{B}_1, \mathcal{B}_2)$ be the sheaves corresponding (a)-(e). Set $W := \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$. Then we observe that $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2) \simeq W^*$ holds with an irreducible $\mathrm{SL}(V)$ -module W in each case. Therefore $\mathrm{Hom}(W, W) \simeq \mathbb{C}$, and the $\mathrm{SL}(V)$ -equivariant map $\mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{B}_1 \otimes \mathcal{B}_2$ are unique for all.

6.2. Closed subschemes $\Delta_3^{\mathcal{Z}}$ in $\mathcal{Z}_3 \times \mathcal{X}$ and $\Delta_2^{\mathcal{Z}}$ in $\mathcal{Z}_2 \times \mathcal{X}$.

In this section, we give preliminary constructions for the proof of Theorem 6.1.1 and present a sketch of the proof. In Proposition 3.1.2, we have found a family of curves on $Y \subset \widetilde{\mathcal{Y}}$ parameterized by $X \subset \mathcal{X}$. The subscheme Δ in $\widetilde{\mathcal{Y}} \times \mathcal{X}$ is the scheme behind this family. We start with the flag variety $\Delta_0 := F(2, 3, V)$, namely, the closed set

$$(6.7) \quad \Delta_0 = \{([V_3], [V_2]) \mid V_2 \subset V_3\} \subset G(3, V) \times G(2, V)$$

with reduced scheme structure. Let us recall that we have defined the family of curves starting from the plane $G_x = \{[\Pi] \in G(3, V) \mid l_x \subset \mathbb{P}(\Pi)\}$ for a line l_x . Note that the plane G_x is nothing but the fiber of $\Delta_0 \rightarrow G(2, V)$ over $[l_x]$.

6.2.1. $\Delta_3^{\mathcal{Z}}$ in $\mathcal{Z}_3 \times \mathcal{X}$.

As we remarked just above Proposition 5.2.1, \mathcal{Z}_3 is contained in $G(3, V) \times \mathcal{Y}_3$. Therefore we obtain a natural morphism $\mathcal{Z}_3 \rightarrow G(3, V)$. By this morphism $\mathcal{Z}_3 \rightarrow G(3, V)$ and $g: \mathcal{X} \rightarrow G(2, V)$, we have the product morphism $\mathcal{Z}_3 \times \mathcal{X} \rightarrow G(3, V) \times G(2, V)$. Define the subset $\Delta_3^{\mathcal{Z}}$ in $\mathcal{Z}_3 \times \mathcal{X}$ as the pull-back of Δ_0 by this morphism.

Let $\Delta_G \subset G(2, T(-1)) \times \mathcal{X}$ be the pull-back of Δ_0 by the morphism $G(2, T(-1)) \times \mathcal{X} \rightarrow G(3, V) \times G(2, V)$. Then $\Delta_3^{\mathcal{Z}}$ is the pull-back of Δ_G . $\Delta_3^{\mathcal{Z}}$ is a $G(2, 5)$ -bundle over Δ_G since so is \mathcal{Z}_3 over $G(2, T(-1))$ by Proposition 5.2.5.

For any point $x \in \mathcal{X}$, we denote by $\Delta_{3,x}^{\mathcal{Z}}$ the fiber of $\Delta_3^{\mathcal{Z}} \rightarrow \mathcal{X}$ over x . Let l_x be the line in $\mathbb{P}(V)$ corresponding to $g(x) \in G(2, V)$.

Proposition 6.2.1. *For any point $x \in \mathcal{X}$, the variety $\Delta_{3,x}^{\mathcal{Z}}$ has a structure of a $G(2, 5)$ -bundle over the blow-up of $\mathbb{P}(V)$ along l_x . In particular, $\Delta_{3,x}^{\mathcal{Z}}$ is and hence $\Delta_3^{\mathcal{Z}}$ is smooth.*

Proof. Let $\Delta_{G,x}$ be the fiber of $\Delta_G \rightarrow \mathcal{X}$ over x . By the above discussion, $\Delta_{3,x}^{\mathcal{Z}}$ is a $G(2, 5)$ -bundle over $\Delta_{G,x}$. Therefore we have only to show that $\Delta_{G,x}$ is isomorphic to the blow-up of $\mathbb{P}(V)$ along l_x .

We describe $G(2, T(-1))$ as the universal family of planes on $\mathbb{P}(V)$, namely,

$$G(2, T(-1)) = \{([V_3], [V_1]) \mid V_1 \subset V_3\} \subset G(3, V) \times \mathbb{P}(V).$$

Then it holds that

$$\Delta_{G,x} = \{([V_3], [V_1]) \mid V_1 \subset V_3, l_x \subset \mathbb{P}(V_3)\} \subset G(2, T(-1)).$$

It turns out that the natural projection morphism from $\Delta_{G,x}$ to $\mathbb{P}(V)$ sending $([V_3], [V_1])$ to $[V_1]$ is the blow-up of $\mathbb{P}(V)$ along the line l_x . \square

Let $\tilde{\pi}_{\mathcal{Z}_3}$ be the morphism $\pi_{\mathcal{Z}_3} \times \text{id}_{\mathcal{X}}: \mathcal{Z}_3 \times \mathcal{X} \rightarrow \mathcal{Y}_3 \times \mathcal{X}$ and set $\Delta_3 := \tilde{\pi}_{\mathcal{Z}_3}(\Delta_3^{\mathcal{Z}})$ with reduced structure. Then we have

Proposition 6.2.2. *Let $[V_1]$ be a point of $\mathbb{P}(V)$. If $[V_1] \notin l_x$, then the fiber of $\Delta_{3,x} \rightarrow \mathbb{P}(V)$ is isomorphic to $G(2, 5)$. If $[V_1] \in l_x$, then the fiber of $\Delta_{3,x} \rightarrow \mathbb{P}(V)$ is isomorphic to the 8-dimensional Schubert cycle $\{[U_3] \mid \mathbb{P}(U_3) \cap \Pi \neq \emptyset\} \subset G(3, \Lambda^2 V/V_1)$, where Π is a fixed plane of $\mathbb{P}(\Lambda^2 V/V_1)$. In particular, $\Delta_{3,x}^{\mathcal{Z}} \rightarrow \Delta_{3,x}$ is and hence $\Delta_3^{\mathcal{Z}} \rightarrow \Delta_3$ is birational.*

Proof. Since $\Delta_{3,x}^{\mathcal{Z}}$ is the pull-back of $\Delta_{G,x}$, we have

$$(6.8) \quad \Delta_{3,x}^{\mathcal{Z}} = \left\{ ([V_3]; [V_1], [U_3]) \mid \begin{array}{l} V_1 \subset V_3, l_x \subset \mathbb{P}(V_3), \\ U_3 \subset \Lambda^2(V/V_1), [\Lambda^2(V_3/V_1)] \in \mathbb{P}(U_3) \end{array} \right\} \subset \mathcal{Z}_3,$$

Therefore we have

$$(6.9) \quad \Delta_{3,x} = \left\{ ([V_1], [U_3]) \mid \begin{array}{l} U_3 \subset \Lambda^2(V/V_1), \exists V_3 \text{ s.t.} \\ V_1 \subset V_3, [\Lambda^2(V_3/V_1)] \in \mathbb{P}(U_3), l_x \subset \mathbb{P}(V_3) \end{array} \right\} \subset \mathcal{Y}_3$$

If $[V_1] \notin l_x$, then V_3 in the description of $\Delta_{3,x}$ is uniquely determined as $V_3 = V_1 \oplus \mathbb{C}l_x$. Therefore the fiber of $\Delta_{3,x} \rightarrow \mathbb{P}(V)$ over $[V_1]$ is isomorphic to $G(2, 5)$. In particular the natural morphism $\Delta_{3,x}^{\mathcal{Z}} \rightarrow \Delta_{3,x}$ is, and hence $\Delta_3^{\mathcal{Z}} \rightarrow \Delta_3$ is birational. Assume that $[V_1] \in l_x$. Then V_3/V_1 's in (6.9) form a 2-plane in $G(2, V/V_1)$; $\{[V_3/V_1] \mid \mathbb{C}l_x/V_1 \subset V_3/V_1\}$, which we denote by Π_x . Note that conics in Π_x are ρ -conics. Then the fiber of $\Delta_{3,x} \rightarrow \mathbb{P}(V)$ over $[V_1]$ is the Schubert cycle; $\{[U_3] \mid \mathbb{P}(U_3) \cap \Pi_x \neq \emptyset\} \subset G(3, \Lambda^2 V/V_1)$. \square

Remark. The Schubert cycle $\{[U_3] \mid \mathbb{P}(U_3) \cap \Pi_x \neq \emptyset\}$ has a natural resolution of singularities; $\{([V_3], [U_3]) \mid [V_3/V_1] \in \mathbb{P}(U_3), l_x \subset \mathbb{P}(V_3)\}$, which is contained in $\Delta_{3,x}^{\mathcal{Z}}$ and has a $G(2, 5)$ -bundle structure over $\{[V_3] \mid l_x \subset \mathbb{P}(V_3)\} \simeq \mathbb{P}^2$.

6.2.2. $\Delta_2^{\mathcal{Z}}$ in $\mathcal{Z}_2 \times \mathcal{X}$.

Let $\bar{\rho}: \mathcal{Z}_2 \rightarrow G(3, V)$ be the composite of $\mathcal{Z}_2 \rightarrow \mathcal{Z}_3$ and $\mathcal{Z}_3 \rightarrow G(3, V)$. Define $\Delta_2^{\mathcal{Z}}$ to be the pull back of Δ_0 to $\mathcal{Z}_2 \times \mathcal{X}$ by $\bar{\rho} \times g: \mathcal{Z}_2 \times \mathcal{X} \rightarrow G(3, V) \times G(2, V)$. Then $\Delta_2^{\mathcal{Z}}$ is also the pull back of $\Delta_3^{\mathcal{Z}}$. We denote by $\Delta_{2,x}^{\mathcal{Z}}$ the fiber of $\Delta_2^{\mathcal{Z}} \rightarrow \mathcal{X}$ over x .

Note that $\mathcal{Z}_\rho \cap \Delta_{3,x}^{\mathcal{Z}}$ is smooth since it is a \mathbb{P}^1 -bundle over $\Delta_{G,x}$ by Proposition 5.2.6. Since $\mathcal{Z}_2 \rightarrow \mathcal{Z}_3$ is the blow-up along $\mathcal{Z}_\rho = \pi_{\mathcal{Z}_3}^{-1}(\mathcal{P}_\rho)$, we have the following:

Proposition 6.2.3. $\Delta_{2,x}^{\mathcal{Z}}$ is the blow-up of $\Delta_{3,x}^{\mathcal{Z}}$ along $\mathcal{Z}_\rho \cap \Delta_{3,x}^{\mathcal{Z}}$. In particular, $\Delta_{2,x}^{\mathcal{Z}}$ is and hence $\Delta_2^{\mathcal{Z}}$ is smooth.

Our proof of Theorem 6.1.1 will be presented in Subsections 6.3–6.5.

In Subsection 6.3, we construct a locally free resolution of the ideal sheaf $\mathcal{I}_2^{\mathcal{Z}}$ of $\Delta_2^{\mathcal{Z}}$ in $\mathcal{Z}_2 \times \mathcal{X}$ from that of the ideal sheaf \mathcal{I}_0 of Δ_0 in $G(3, V) \times G(2, V)$ (see (6.10) and (6.11)).

For notational simplicity, we write the transforms of $\mathcal{P}_\sigma \subset \mathcal{Y}_3$ on \mathcal{Y}_2 and $\widetilde{\mathcal{Y}}$ by the same \mathcal{P}_σ . We set $\mathcal{Z}_2^\circ := \mathcal{Z}_2 \setminus \pi_{\mathcal{Z}_2}^{-1}(\mathcal{P}_\sigma)$ and $\mathcal{Y}_2^\circ := \mathcal{Y}_2 \setminus \mathcal{P}_\sigma$. As we see in Subsection 5.3, $\mathcal{Z}_2^\circ \rightarrow \mathcal{Y}_2^\circ$ is a conic bundle and a fiber of $\tilde{\pi}_{\mathcal{Z}_2}$ is a conic on $G(3, V)$ which is not a σ -conic. Then we calculate the pushforward of the locally free resolution of $\mathcal{I}_2^{\mathcal{Z}}$ by $\tilde{\pi}_{\mathcal{Z}_2} := \pi_{\mathcal{Z}_2} \times \text{id}_{\mathcal{X}}: \mathcal{Z}_2 \times \mathcal{X} \rightarrow \mathcal{Y}_2 \times \mathcal{X}$ over the flat locus $\mathcal{Y}_2^\circ \times \mathcal{X}$. Then, also in Subsection 6.3, we obtain a closed subscheme Δ_2° of \mathcal{Y}_2° and a locally free resolution of its ideal sheaf \mathcal{I}_2° (see (6.16), Proposition 6.3.7 and (6.22)).

In Subsection 6.4, we calculate the pushforward of the locally free resolution of \mathcal{I}_2° by the morphism $\mathcal{Y}_2^\circ \times \mathcal{X} \rightarrow \widetilde{\mathcal{Y}}^\circ \times \mathcal{X}$, where we set $\widetilde{\mathcal{Y}}^\circ := \widetilde{\mathcal{Y}} \setminus \mathcal{P}_\sigma$. Then we obtain an ideal sheaf \mathcal{I}° on $\widetilde{\mathcal{Y}}^\circ \times \mathcal{X}$ and its locally free resolution (see (6.34)). Let $\iota_{\widetilde{\mathcal{Y}}^\circ}$ be the open immersion $\widetilde{\mathcal{Y}}^\circ \times \mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}} \times \mathcal{X}$. Finally, we show that the

pushforward $\iota_{\widetilde{\mathcal{Y}} \circ *}$ of \mathcal{I}^o and its locally free resolution coincides with the ideal sheaf \mathcal{I} and its locally free resolution (6.2), respectively as in Theorem 6.1.1.

In Subsection 6.5, we show that the closed subscheme Δ of $\widetilde{\mathcal{Y}} \times \widetilde{\mathcal{X}}$ defined by \mathcal{I} is normal and Cohen-Macaulay.

6.3. A locally free resolution of an ideal sheaf \mathcal{I}_2^o on $\mathcal{Y}_2^o \times \widetilde{\mathcal{X}}$.

Starting with the following locally free resolution of the ideal sheaf of the subscheme $\Delta_0 \subset \mathbf{G}(3, V) \times \mathbf{G}(2, V)$ as in (6.7), we obtain the resolution of the ideal sheaf \mathcal{I}_2^o on $\mathcal{Y}_2^o \times \widetilde{\mathcal{X}}$. Our construction consists of four steps.

Proposition 6.3.1. *The ideal sheaf \mathcal{I}_0 of Δ_0 in $\mathbf{G}(3, V) \times \mathbf{G}(2, V)$ has the following Koszul resolution:*

$$(6.10) \quad 0 \rightarrow \wedge^4(\mathcal{W}^* \boxtimes \mathcal{F}^*) \rightarrow \wedge^3(\mathcal{W}^* \boxtimes \mathcal{F}^*) \rightarrow \wedge^2(\mathcal{W}^* \boxtimes \mathcal{F}^*) \rightarrow \mathcal{W}^* \boxtimes \mathcal{F}^* \rightarrow \mathcal{I}_0 \rightarrow 0.$$

Proof. Tensoring the two natural surjections $V \otimes \mathcal{O}_{\mathbf{G}(3, V)} \rightarrow \mathcal{W}$ and $V^* \otimes \mathcal{O}_{\mathbf{G}(2, V)} \rightarrow \mathcal{F}$, we obtain a map $(V \otimes V^*) \otimes \mathcal{O}_{\mathbf{G}(3, V) \times \mathbf{G}(2, V)} \rightarrow \mathcal{W} \boxtimes \mathcal{F}$. Associated to the identity in $\mathrm{Hom}(V, V) \simeq V \otimes V^*$, we obtain the map $\mathcal{O}_{\mathbf{G}(3, V) \times \mathbf{G}(2, V)} \rightarrow \mathcal{W} \boxtimes \mathcal{F}$. We show that Δ_0 is the scheme of zeros of the section associated to this map. Indeed, at a point $([V_3], [V_2])$ of $\mathbf{G}(3, V) \times \mathbf{G}(2, V)$, the fiber of \mathcal{W} is V/V_3 and the fiber of \mathcal{F} is V_2^* . Then it is easy to see that the identity in $V \otimes V^*$ is contained in the kernel of the natural map $V \otimes V^* \rightarrow (V/V_3) \otimes V_2^*$ if and only if $V_2 \subset V_3$, namely, $([V_3], [V_2]) \in \Delta_0$. \square

Step 1. Let $\mathcal{I}_2^{\mathcal{F}}$ be the ideal sheaf of $\Delta_2^{\mathcal{F}}$ on $\mathcal{X}_2 \times \widetilde{\mathcal{X}}$. By pulling back (6.10) to $\mathcal{X}_2 \times \widetilde{\mathcal{X}}$, we see that $\mathcal{I}_2^{\mathcal{F}}$ has the following locally free resolution:

$$(6.11) \quad \begin{aligned} 0 \rightarrow \wedge^4(\bar{\rho}^* \mathcal{W}^* \boxtimes g^* \mathcal{F}^*) &\rightarrow \wedge^3(\bar{\rho}^* \mathcal{W}^* \boxtimes g^* \mathcal{F}^*) \rightarrow \\ &\wedge^2(\bar{\rho}^* \mathcal{W}^* \boxtimes g^* \mathcal{F}^*) \rightarrow \bar{\rho}^* \mathcal{W}^* \boxtimes g^* \mathcal{F}^* \rightarrow \mathcal{I}_2^{\mathcal{F}} \rightarrow 0. \end{aligned}$$

In this step, we calculate the pushforward of this by $\tilde{\pi}_{\mathcal{X}_2}: \mathcal{X}_2 \times \widetilde{\mathcal{X}} \rightarrow \mathcal{Y}_2 \times \widetilde{\mathcal{X}}$. See Proposition 6.3.4.

Until the end of this subsection 6.3, we consider only on $\mathcal{Y}_2^o \times \widetilde{\mathcal{X}}$ to calculate the higher direct images for $\tilde{\pi}_{\mathcal{X}_2}$. To simplify the notation, we abbreviate the symbols for the restriction.

For our calculations below, we prepare the following two lemmas:

Lemma 6.3.2. *Let $P = P_{V_2}$ be the ρ -plane in $\mathbf{G}(3, V)$ associated to some two dimensional vector space V_2 in V (cf. Subsection 5.1). Then $\mathcal{W}|_P \simeq T_P(-1)$.*

Proof. From the natural surjection $V \otimes \mathcal{O}_{\mathbf{G}(3, V)} \rightarrow \mathcal{W}$, we obtain the surjection $V/V_2 \otimes \mathcal{O}_P \rightarrow \mathcal{W}|_P$. Since $P \simeq \mathbb{P}(V/V_2)$, this surjection is contained in the Euler sequence of P . Therefore $\mathcal{W}|_P \simeq T_P(-1)$. \square

Lemma 6.3.3. *Let q be a conic on $\mathbf{G}(3, V)$. Suppose that q is not a σ -conic. Then $H^\bullet(\mathcal{W}^*|_q) = 0$. Moreover, if q is smooth, then $\mathcal{W}|_q \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$.*

Proof. We denote by \mathbb{P}_q^2 the plane spanned by q .

First we suppose q is a τ -conic. As we have shown in [HoTa3, Subsect.5.4], there exists an $S \simeq \mathbf{G}(2, 4) \subset \mathbf{G}(3, V)$ such that $q \subset S$. The conic q is a complete

intersection in S since $\mathbb{P}_q^2 \cap S = q$, and then \mathcal{O}_q has the following Koszul resolution as a \mathcal{O}_S -module:

$$0 \rightarrow \mathcal{O}_S(-3) \rightarrow \mathcal{O}_S(-2)^{\oplus 3} \rightarrow \mathcal{O}_S(-1)^{\oplus 3} \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_q \rightarrow 0.$$

Tensoring this exact sequence with $\mathcal{W}^*|_S$ and using Theorem 2.1.1, it is easy to derive $H^\bullet(\mathcal{W}^*|_q) = 0$.

Second we suppose q is a ρ -conic. By Lemma 6.3.2, it holds that $\mathcal{W}|_{\mathbb{P}_q^2} \simeq T_{\mathbb{P}_q^2}(-1)$. Tensoring $\mathcal{W}^*|_{\mathbb{P}_q^2}$ with the exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}_q^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}_q^2} \rightarrow \mathcal{O}_q \rightarrow 0$, we have

$$0 \rightarrow \Omega_{\mathbb{P}_q^2}^1(-1) \rightarrow \Omega_{\mathbb{P}_q^2}^1(1) \rightarrow \mathcal{W}^*|_q \rightarrow 0.$$

By Theorem 2.1.1, it holds that all the cohomology groups of $\Omega_{\mathbb{P}_q^2}^1(-1)$ and $\Omega_{\mathbb{P}_q^2}^1(1)$ vanish. Thus we have $H^\bullet(\mathcal{W}^*|_q) = 0$.

The last part follows from the first. \square

The following argument to obtain (6.16) is inspired by [Ku2, Lemma 8.2]. Let us split (6.11) into the short exact sequences:

$$(6.12) \quad 0 \rightarrow \wedge^4(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow \wedge^3(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow \mathcal{K}_1 \rightarrow 0.$$

$$(6.13) \quad 0 \rightarrow \mathcal{K}_1 \rightarrow \wedge^2(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow \mathcal{K}_2 \rightarrow 0.$$

$$(6.14) \quad 0 \rightarrow \mathcal{K}_2 \rightarrow \bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^* \rightarrow \mathcal{I}_2^{\mathcal{Z}} \rightarrow 0.$$

By (6.14) and Lemma 6.3.3, we have

$$(6.15) \quad \tilde{\pi}_{\mathcal{Z}_2*}(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) = R^1\tilde{\pi}_{\mathcal{Z}_2*}(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) = 0.$$

Hence we have $\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{I}_2^{\mathcal{Z}} = R^1\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{K}_2$, and $\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{K}_2 = 0$. Thus by (6.13),

$$0 \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{K}_1 \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^2(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{K}_2 = \tilde{\pi}_{\mathcal{Z}_2*}\mathcal{I}_2^{\mathcal{Z}} \rightarrow 0.$$

Since $\wedge^2(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*)|_q \simeq \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 6}$ on a smooth fiber q by Lemma 6.3.3 and $\tilde{\pi}_{\mathcal{Z}_2*}\wedge^2(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*)$ is torsion free, we have $\tilde{\pi}_{\mathcal{Z}_2*}\wedge^2(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) = 0$. This implies that $\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{K}_1 = 0$ by (6.13). Thus by (6.12),

$$0 \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^4(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^3(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{K}_1 \rightarrow 0.$$

Therefore we obtain the following result:

Proposition 6.3.4. *Define $\mathcal{I}_2^{\mathcal{O}} := \tilde{\pi}_{\mathcal{Z}_2*}\mathcal{I}_2^{\mathcal{Z}}$, which is an ideal sheaf on $\mathcal{Z}_2^{\mathcal{O}} \times \mathcal{X}$. Then there exists an exact sequence on the locus $\mathcal{Z}_2^{\mathcal{O}} \times \mathcal{X}$:*

$$(6.16) \quad 0 \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^4(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^3(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^2(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \rightarrow \mathcal{I}_2^{\mathcal{O}} \rightarrow 0.$$

Let $\Delta_2^{\mathcal{Z}^{\mathcal{O}}}$ and $\Delta_2^{\mathcal{O}}$ be the closed subschemes of $\mathcal{Z}_2^{\mathcal{O}} \times \mathcal{X}$ and $\mathcal{Z}_2^{\mathcal{O}} \times \mathcal{X}$ defined by $\mathcal{I}_2^{\mathcal{Z}^{\mathcal{O}}}$ and $\mathcal{I}_2^{\mathcal{O}}$ respectively. We set $\pi_{\Delta} := \tilde{\pi}_{\mathcal{Z}_2}|_{\Delta_2^{\mathcal{Z}^{\mathcal{O}}}}$. To show Δ is normal and Cohen-Macaulay in Subsection 6.5, we prepare the following Lemma:

Lemma 6.3.5. $\pi_{\Delta*}\mathcal{O}_{\Delta_2^{\mathcal{Z}^{\mathcal{O}}}} = \mathcal{O}_{\Delta_2^{\mathcal{O}}}$ and $R^1\pi_{\Delta*}\mathcal{O}_{\Delta_2^{\mathcal{Z}^{\mathcal{O}}}} = 0$.

Proof. By (6.14) and (6.15), we have $R^1\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{I}_2^{\mathcal{Z}^{\mathcal{O}}} = 0$. Taking the higher direct image of the exact sequence $0 \rightarrow \mathcal{I}_2^{\mathcal{Z}^{\mathcal{O}}} \rightarrow \mathcal{O}_{\mathcal{Z}_2^{\mathcal{O}} \times \mathcal{X}} \rightarrow \mathcal{O}_{\Delta_2^{\mathcal{Z}^{\mathcal{O}}}} \rightarrow 0$, we obtain the exact sequence $0 \rightarrow \mathcal{I}_2^{\mathcal{O}} \rightarrow \mathcal{O}_{\mathcal{Z}_2^{\mathcal{O}} \times \mathcal{X}} \rightarrow \pi_{\Delta*}\mathcal{O}_{\Delta_2^{\mathcal{Z}^{\mathcal{O}}}} \rightarrow 0$ and $R^1\pi_{\Delta*}\mathcal{O}_{\Delta_2^{\mathcal{Z}^{\mathcal{O}}}} = 0$ since $R^1\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{I}_2^{\mathcal{Z}^{\mathcal{O}}} = 0$ and $R^1\tilde{\pi}_{\mathcal{Z}_2*}\mathcal{O}_{\mathcal{Z}_2^{\mathcal{O}} \times \mathcal{X}} = 0$. Thus $\pi_{\Delta*}\mathcal{O}_{\Delta_2^{\mathcal{Z}^{\mathcal{O}}}} = \mathcal{O}_{\Delta_2^{\mathcal{O}}}$. \square

Step 2. In this step, we rewrite each term of the resolution (6.16) by the Grothendieck-Verdier duality. See Proposition 6.3.7.

Lemma 6.3.6. *Let q be a conic on $G(3, V)$. Suppose that q is not a σ -conic. Then $H^1(q, \wedge^i \mathcal{W}^{\oplus 2} \otimes \mathcal{O}_q(-1)) = 0$ ($1 \leq i \leq 4$).*

Proof. First suppose that q is a τ -conic. As in the proof of Claim 6.3.3, take $S \simeq G(2, 4) \subset G(3, V)$ such that $q \subset S$ and consider the Koszul resolution of \mathcal{O}_q on S . Tensoring this exact sequence with $\wedge^i \mathcal{W}|_S^{\oplus 2} \otimes \mathcal{O}_S(-1)$, we obtain

$$0 \rightarrow (\wedge^i \mathcal{W}|_S^{\oplus 2})(-4) \rightarrow (\wedge^i \mathcal{W}|_S^{\oplus 2})(-3)^{\oplus 3} \rightarrow (\wedge^i \mathcal{W}|_S^{\oplus 2})(-2)^{\oplus 3} \rightarrow \wedge^i \mathcal{W}|_S^{\oplus 2}(-1) \rightarrow \wedge^i \mathcal{W}^{\oplus 2} \otimes \mathcal{O}_q(-1) \rightarrow 0.$$

Using Theorem 2.1.1, it is easy to derive the assertion.

Second suppose that q is a ρ -conic. As in the proof of Lemma 6.3.3, tensoring $(\wedge^i \mathcal{W}^{\oplus 2}|_{\mathbb{P}_q^2})(-1) \simeq (\wedge^i T_{\mathbb{P}_q^2}(-1)^{\oplus 2})(-1)$ with the exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}_q^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}_q^2} \rightarrow \mathcal{O}_q \rightarrow 0$, we have

$$0 \rightarrow (\wedge^i T_{\mathbb{P}_q^2}(-1)^{\oplus 2})(-3) \rightarrow (\wedge^i T_{\mathbb{P}_q^2}(-1)^{\oplus 2})(-1) \rightarrow \wedge^i \mathcal{W}^{\oplus 2} \otimes \mathcal{O}_q(-1) \rightarrow 0.$$

Computing all the cohomology groups of $(\wedge^i T_{\mathbb{P}_q^2}(-1)^{\oplus 2})(-3)$ and $(\wedge^i T_{\mathbb{P}_q^2}(-1)^{\oplus 2})(-1)$ by Theorem 2.1.1, we have the assertion. \square

By Lemma 6.3.6, we may apply the latter part of the Grothendieck-Verdier duality 2.1.2 to the morphism $\mathcal{X}_2^o \times \mathcal{X} \rightarrow \mathcal{Y}_2^o \times \mathcal{X}$, and then we have

$$R^1 \tilde{\pi}_{\mathcal{X}_2*} \wedge^i (\bar{\rho}^* \mathcal{W}^* \boxtimes g^* \mathcal{F}^*) \simeq \left(\tilde{\pi}_{\mathcal{X}_2*} \{ \wedge^i (\bar{\rho}^* \mathcal{W} \boxtimes g^* \mathcal{F}) \otimes \omega_{\mathcal{X}_2 \times \mathcal{X} / \mathcal{Y}_2 \times \mathcal{X}} \} \right)^*.$$

Note that $\omega_{\mathcal{X}_2 \times \mathcal{X} / \mathcal{Y}_2 \times \mathcal{X}} = \text{pr}_2^* \omega_{\mathcal{X}_2 / \mathcal{Y}_2} = \omega_{\mathcal{X}_2 / \mathcal{Y}_2} \boxtimes \mathcal{O}_{\mathcal{X}}$. By Proposition 5.3.1, we have $K_{\mathcal{X}_2 / \mathcal{Y}_2} = M_{\mathcal{X}_2} - N_{\mathcal{X}_2}$. Thus we have

$$(6.17) \quad R^1 \tilde{\pi}_{\mathcal{X}_2*} \wedge^i (\bar{\rho}^* \mathcal{W}^* \boxtimes g^* \mathcal{F}^*) \simeq \left(\tilde{\pi}_{\mathcal{X}_2*} \{ \wedge^i (\bar{\rho}^* \mathcal{W} \boxtimes g^* \mathcal{F}) \otimes (\mathcal{O}_{\mathcal{X}_2}(-N_{\mathcal{X}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}) \} \otimes (\mathcal{O}_{\mathcal{Y}_2}(M_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}) \right)^*.$$

We write down this more explicitly. For this, we use the following formula (see [FH, Exercise 6.11]):

$$(6.18) \quad \wedge^i (\bar{\rho}^* \mathcal{W} \boxtimes g^* \mathcal{F}) \simeq \bigoplus_{\lambda} \Sigma^{\lambda} \bar{\rho}^* \mathcal{W} \boxtimes \Sigma^{\lambda'} g^* \mathcal{F},$$

where λ are partitions of i with at most 2 rows and column, and λ' is the partitions dual to λ .

Proposition 6.3.7. *The exact sequence (6.16) $\otimes \mathcal{O}_{\mathcal{Y}_2}(M_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}(2L_{\mathcal{X}})$ on the locus $\mathcal{Y}_2^o \times \mathcal{X}$ is presented as follows:*

$$\begin{aligned} 0 \rightarrow (\pi_{\mathcal{X}_2*} \mathcal{O}_{\mathcal{X}_2}(N_{\mathcal{X}_2}))^* \boxtimes \mathcal{O}_{\mathcal{X}} \rightarrow \{ \pi_{\mathcal{X}_2*} (\bar{\rho}^* \mathcal{W}) \}^* \boxtimes g^* \mathcal{F} \rightarrow \\ \mathcal{O}_{\mathcal{Y}_2} \boxtimes g^* \mathcal{S}^2 \mathcal{F} \oplus \{ \pi_{\mathcal{X}_2*} (\{ \bar{\rho}^* \mathcal{S}^2 \mathcal{W} \}(-N_{\mathcal{X}_2})) \}^* \boxtimes \mathcal{O}_{\mathcal{X}}(L_{\mathcal{X}}) \rightarrow \\ \mathcal{I}_2^o \otimes \mathcal{O}_{\mathcal{Y}_2}(M_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}(2L_{\mathcal{X}}) \rightarrow 0 \end{aligned}$$

Proof. By using (6.18), we evaluate $\wedge^4 (\bar{\rho}^* \mathcal{W} \boxtimes g^* \mathcal{F}) = \mathcal{O}_{\mathcal{X}_2}(2N_{\mathcal{X}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}(2L_{\mathcal{X}})$. Then we have

$$R^1 \tilde{\pi}_{\mathcal{X}_2*} \wedge^4 (\bar{\rho}^* \mathcal{W}^* \boxtimes g^* \mathcal{F}^*) \simeq (\{ \pi_{\mathcal{X}_2*} \mathcal{O}_{\mathcal{X}_2}(N_{\mathcal{X}_2}) \}^* \otimes \mathcal{O}_{\mathcal{Y}_2}(-M_{\mathcal{Y}_2})) \boxtimes \mathcal{O}_{\mathcal{X}}(-2L_{\mathcal{X}}).$$

Similarly, we evaluate $\wedge^3(\bar{\rho}^*\mathcal{W} \boxtimes g^*\mathcal{F}) \simeq \bar{\rho}^*\mathcal{W}(N_{\mathcal{Z}_2}) \boxtimes g^*\mathcal{F}(L_{\mathcal{X}})$ and have

$$R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^3(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*) \simeq \{\pi_{\mathcal{Z}_2*}(\bar{\rho}^*\mathcal{W})^* \otimes \mathcal{O}_{\mathcal{Y}_2}(-M_{\mathcal{Y}_2})\} \boxtimes (g^*\mathcal{F}^*(-L_{\mathcal{X}})).$$

Finally, we have

$$\begin{aligned} \wedge^2(\bar{\rho}^*\mathcal{W} \boxtimes g^*\mathcal{F}) &\simeq \wedge^2\bar{\rho}^*\mathcal{W} \boxtimes S^2(g^*\mathcal{F}) \oplus S^2(\bar{\rho}^*\mathcal{W}) \boxtimes \wedge^2g^*\mathcal{F} \\ &\simeq \mathcal{O}_{\mathcal{Z}_3}(N_{\mathcal{Z}_3}) \boxtimes g^*S^2\mathcal{F} \oplus \bar{\rho}^*S^2\mathcal{W} \boxtimes \mathcal{O}_{\mathcal{X}}(L_{\mathcal{X}}). \end{aligned}$$

Using this we evaluate $R^1\tilde{\pi}_{\mathcal{Z}_2*}\wedge^2(\bar{\rho}^*\mathcal{W}^* \boxtimes g^*\mathcal{F}^*)$ as

$$\mathcal{O}_{\mathcal{Y}_2}(-M_{\mathcal{Y}_2}) \boxtimes g^*S^2\mathcal{F} \oplus \{\pi_{\mathcal{Z}_2*}(\{\bar{\rho}^*S^2\mathcal{W}\}(-N_{\mathcal{Z}_2}))^* \otimes \mathcal{O}_{\mathcal{Y}_2}(-M_{\mathcal{Y}_2})\} \boxtimes \mathcal{O}_{\mathcal{X}}(-L_{\mathcal{X}}).$$

□

In the following two steps, we will characterize the following sheaves:

$$(6.19) \quad \pi_{\mathcal{Z}_2*}\mathcal{O}_{\mathcal{Z}_2}(N_{\mathcal{Z}_2}), \pi_{\mathcal{Z}_2*}(\bar{\rho}^*\mathcal{W}), \pi_{\mathcal{Z}_2*}(\{\bar{\rho}^*S^2\mathcal{W}\}(-N_{\mathcal{Z}_2})),$$

which appear in the resolution. Here we present a preliminary result.

Lemma 6.3.8. *All the sheaves in (6.19) are locally free on \mathcal{Y}_2° .*

Proof. By the Grauert theorem, it suffices to check that the dimensions of H^0 -terms of the restrictions of the sheaves to fibers in (6.19) are constant on \mathcal{Y}_2° . Let q be the fiber of $\mathcal{Z}_2 \rightarrow \mathcal{Y}_2$ over a point of \mathcal{Y}_2° . Let $[V_1] \in \mathbb{P}(V)$ be the image of q by $\mathcal{Z}_2 \rightarrow \mathbb{P}(V)$. We may consider q as a conic on $G(2, V/V_1)$. Let \mathbb{P}_q^2 be the plane spanned by q .

For $\pi_{\mathcal{Z}_2*}\mathcal{O}_{\mathcal{Z}_2}(N_{\mathcal{Z}_2})$, we have

$$H^0(q, N_{\mathcal{Z}_2}|_q) \simeq H^0(q, \mathcal{O}_q(1)) \simeq H^0(\mathbb{P}_q^2, \mathcal{O}_{\mathbb{P}_q^2}(1)) \simeq \mathbb{C}^3.$$

For $\pi_{\mathcal{Z}_2*}(\bar{\rho}^*\mathcal{W})$, the sheaf $\bar{\rho}^*\mathcal{W}|_q$ is generated by global sections and its degree is two. Therefore, by the Riemann-Roch theorem,

$$H^0(q, \bar{\rho}^*\mathcal{W}|_q) \simeq \mathbb{C}^4.$$

For $\pi_{\mathcal{Z}_2*}(\{\bar{\rho}^*S^2\mathcal{W}\}(-N_{\mathcal{Z}_2}))$, we can show $H^0(q, (\{\bar{\rho}^*S^2\mathcal{W}\}(-N_{\mathcal{Z}_2}))|_q) \simeq \mathbb{C}^3$ by similar computations to those in the proof of Lemma 6.3.6. Here we do the calculations only in the case where q is a ρ -conic. Tensoring $(\{\bar{\rho}^*S^2\mathcal{W}\}(-N_{\mathcal{Z}_2}))|_{\mathbb{P}_q^2} \simeq (S^2T_{\mathbb{P}_q^2}(-1))(-1)$ with the exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}_q^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}_q^2} \rightarrow \mathcal{O}_q \rightarrow 0$, we have

$$0 \rightarrow (S^2T_{\mathbb{P}_q^2}(-1))(-3) \rightarrow (S^2T_{\mathbb{P}_q^2}(-1))(-1) \rightarrow (S^2T_{\mathbb{P}_q^2}(-1))(-1)|_q \rightarrow 0.$$

We can compute all the cohomology groups of $(S^2T_{\mathbb{P}_q^2}(-1))(-3)$ and $(S^2T_{\mathbb{P}_q^2}(-1))(-1)$ by Theorem 2.1.1 as follows: all the cohomology groups of $(S^2T_{\mathbb{P}_q^2}(-1))(-1)$ vanish. The only nonvanishing cohomology group of $(S^2T_{\mathbb{P}_q^2}(-1))(-3)$ is H^1 and

$$(6.20) \quad H^1(\mathbb{P}_q^2, (S^2T_{\mathbb{P}_q^2}(-1))(-3)) \simeq W^* \otimes \wedge^3 W^*,$$

where W is the three dimensional subspace of $\wedge^2(V/V_1)$ such that $\mathbb{P}_q^2 = \mathbb{P}(W)$. Consequently, we have

$$H^0(q, (\{\bar{\rho}^*S^2\mathcal{W}\}(-N_{\mathcal{Z}_2}))|_q) \simeq H^1(\mathbb{P}_q^2, (S^2T_{\mathbb{P}_q^2}(-1))(-3)) \simeq \mathbb{C}^3.$$

□

Step 3. In this step, we obtain good approximations of the sheaves in (6.19) by considering the corresponding sheaves on \mathcal{Z}_2^t of similar forms. See Lemma 6.3.9. An advantage of \mathcal{Z}_2^t is that its structure sheaf $\mathcal{O}_{\mathcal{Z}_2^t}$ has a nice Koszul resolution as \mathcal{O}_{G_2} -module.

Recall that, in Subsection 5.3, we have identified $\mathcal{Z}_2^u = \mathcal{Z}_3^u \times_{\mathcal{Y}_3} \mathcal{Y}_2$ with the blow-up along $\pi_{\mathcal{Z}_3^u}^{-1}(\mathcal{P}_\rho) \subset \mathcal{Z}_3^u$. The restriction of \mathcal{Z}_2^u to $G_2 = G(2, T(-1)) \times_{\mathbb{P}(V)} \mathcal{Y}_2$ is denoted by \mathcal{Z}_2^t (i.e., $\mathcal{Z}_2^t := G_2 \cap \mathcal{Z}_2^u$). As noted there, \mathcal{Z}_2^t is the total transform of \mathcal{Z}_3 under the blow-up $\mathcal{Z}_3 \times_{\mathcal{Y}_3} \mathcal{Y}_2 \rightarrow \mathcal{Z}_3$ along $\pi_{\mathcal{Z}_3}^{-1}(\mathcal{P}_\rho) \subset \mathcal{Z}_3$ while \mathcal{Z}_2 is the strict transform of \mathcal{Z}_3 .

By Proposition 5.2.3 and $\mathcal{Z}_2^u = \mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*)$, the variety \mathcal{Z}_2^u is a complete intersection in $\mathbb{P}(T(-1)^2) \times_{\mathbb{P}(V)} \mathcal{Y}_2$ with respect to a section of $\mathcal{O}_{\mathbb{P}(T(-1)^2)}(1) \boxtimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}$. Therefore $\mathcal{Z}_2^t = G_2 \cap \mathcal{Z}_2^u$ is the complete intersection in G_2 with respect to a section of $(\mathcal{O}_{\mathbb{P}(T(-1)^2)}(1) \boxtimes \rho_{\mathcal{Y}_2}^* \mathcal{Q})|_{G_2}$. By (4.11) and Proposition 4.2.1, it holds that

$$(\mathcal{O}_{\mathbb{P}(T(-1)^2)}(1) \boxtimes \rho_{\mathcal{Y}_2}^* \mathcal{Q})|_{G_2} \simeq \mathcal{O}_{G(2, T(-1))}(N_{G(2, T(-1))} - L_{G(2, T(-1))}) \boxtimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}.$$

By the description of \mathcal{Z}_2^u , the sheaf $\mathcal{O}_{\mathcal{Z}_2^t}$ has the following Koszul resolution as a \mathcal{O}_{G_2} -module:

$$(6.21) \quad 0 \rightarrow \mathcal{A}_3 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_1 \rightarrow \mathcal{O}_{G_2} \rightarrow \mathcal{O}_{\mathcal{Z}_2^t} \rightarrow 0,$$

where we set

$$\mathcal{A}_i := \mathcal{O}_{G(2, T(-1))}(-iN_{G(2, T(-1))} + iL_{G(2, T(-1))}) \boxtimes \wedge^i \rho_{\mathcal{Y}_2}^* \mathcal{Q}^* \text{ for } i = 0, 1, 2, 3.$$

Let $\pi_{\mathcal{Z}_2^t}: \mathcal{Z}_2^t \rightarrow \mathcal{Y}_2$ and $\bar{\rho}^t: \mathcal{Z}_2^t \rightarrow G(3, V)$ be the natural morphisms, and $N_{\mathcal{Z}_2^t}$ the pull-back of $N_{G(2, T(-1))}$. Using the above Koszul resolution (6.21), we show the isomorphisms (i)–(iii) below. We should note that these isomorphic sheaves provide us good approximations of the sheaves in (6.19). Indeed, for any sheaf \mathcal{B} on \mathcal{Z}_2^u , we have a natural map $\pi_{\mathcal{Z}_2^t*}(\mathcal{B}|_{\mathcal{Z}_2^t}) \rightarrow \pi_{\mathcal{Z}_2*}(\mathcal{B}|_{\mathcal{Z}_2})$ on \mathcal{Y}_2 , which is isomorphic outside F_ρ . Moreover, if $\pi_{\mathcal{Z}_2^t*}(\mathcal{B}|_{\mathcal{Z}_2^t})$ is locally free, then the map is injective. The isomorphisms (i)–(iii) below indicate that this is the case for each sheaf in (6.19).

Lemma 6.3.9. (i) $\pi_{\mathcal{Z}_2^t*} \mathcal{O}_{\mathcal{Z}_2^t}(N_{\mathcal{Z}_2^t}) \simeq \rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$,
(ii) $\pi_{\mathcal{Z}_2^t*}((\bar{\rho}^t)^* \mathcal{W}) \simeq \pi_{\mathcal{Y}_2}^* T(-1)$, and
(iii) $\pi_{\mathcal{Z}_2^t*}(\{(\bar{\rho}^t)^* \mathcal{S}^2 \mathcal{W}\}(-N_{\mathcal{Z}_2^t})) \simeq \rho_{\mathcal{Y}_2}^* \mathcal{Q}(-M_{\mathcal{Y}_2} - F_\rho)$.

Proof. We show (i)–(iii) with aid of Theorem 2.1.1 noting $\text{pr}_2: G_2 \rightarrow \mathcal{Y}_2$ is a $G(2, 4)$ -bundle. Let Γ be a fiber of pr_2 .

To see (i), we tensor (6.21) with $\mathcal{O}_{G_2}(N_{G_2})$, which is the pull-back to G_2 of $\mathcal{O}_{\mathbb{P}(\Omega(1)^2)}(1)|_{G(2, T(-1))}$ by (4.11). It is easy to see that $R^\bullet \text{pr}_{2*}(\mathcal{O}_{G_2}(N_{G_2}) \otimes \mathcal{A}_1) = 0$ for $\bullet > 0$, and $R^\bullet \text{pr}_{2*}(\mathcal{O}_{N_2}(N_{G_2}) \otimes \mathcal{A}_i) = 0$ for $\bullet \geq 0$ and $i = 2, 3$ since $(\mathcal{O}_{G_2}(N_{G_2}) \otimes \mathcal{A}_i)|_\Gamma \simeq \mathcal{O}_\Gamma(-i+1)$. Moreover, $\text{pr}_{2*}(\mathcal{O}_{G_2}(N_{G_2}) \otimes \mathcal{A}_1) = \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(L_{\mathcal{Y}_2})$ and $\text{pr}_{2*} \mathcal{O}_{G_2}(N_{G_2}) = \pi_{\mathcal{Y}_2}^* T(-1)^2$. Therefore we obtain the short exact sequence

$$0 \rightarrow \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(L_{\mathcal{Y}_2}) \rightarrow \pi_{\mathcal{Y}_2}^* T(-1)^2 \rightarrow \pi_{\mathcal{Z}_2^t*} \mathcal{O}_{\mathcal{Z}_2^t}(N_{\mathcal{Z}_2^t}) \rightarrow 0,$$

which coincides with the pull-back of the dual of the universal exact sequence (4.3) twisted by $L_{\mathcal{Y}_2}$ by the proof of Lemma 5.2.2. Thus $\pi_{\mathcal{Z}_2^t*} \mathcal{O}_{\mathcal{Z}_2^t}(N_{\mathcal{Z}_2^t}) \simeq \rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$ as desired.

To see (ii), we tensor (6.21) with the pull back \mathcal{W}_{G_2} on G_2 of \mathcal{W} . We see that $R^\bullet \text{pr}_{2*}(\mathcal{W}_{G_2} \otimes \mathcal{A}_i) = 0$ for $\bullet \geq 0$ and $i = 1, 2, 3$ by Theorem 2.1.1 since $(\mathcal{W}_{G_2} \otimes \mathcal{A}_i)|_\Gamma \simeq \mathcal{W}_\Gamma(-i)$, where \mathcal{W}_Γ is the universal quotient bundle of rank 2 on

$\Gamma \simeq \mathbf{G}(2, 4)$. Moreover, $\mathrm{pr}_{2*}\mathcal{W}_{G_2} \simeq \pi_{\mathcal{Y}_2}^*T(-1)$. Therefore we have $\pi_{\mathcal{X}_2^t*}((\bar{\rho}^t)^*\mathcal{W}) \simeq \pi_{\mathcal{Y}_2}^*T(-1)$ as desired.

To see (iii), we tensor (6.21) with $\mathbf{S}^2\mathcal{W}_{G_2}(-N_{G_2})$. We see that

$$\begin{aligned} R^\bullet \mathrm{pr}_{2*}(\mathbf{S}^2\mathcal{W}_{G_2}(-N_{G_2}) \otimes \mathcal{A}_i) &= 0 \text{ for } \bullet \geq 0 \text{ and } i = 0, 1, 3, \\ R^\bullet \mathrm{pr}_{2*}(\mathbf{S}^2\mathcal{W}_{G_2}(-N_{G_2}) \otimes \mathcal{A}_2) &= 0 \text{ for } \bullet \neq 2 \end{aligned}$$

by Theorem 2.1.1 since $(\mathbf{S}^2\mathcal{W}_{G_2}(-N_{G_2}) \otimes \mathcal{A}_i)|_\Gamma \simeq \mathbf{S}^2\mathcal{W}_\Gamma(-i-1)$. Moreover, by Theorem 2.1.1 again,

$$R^2 \mathrm{pr}_{2*}(\mathbf{S}^2\mathcal{W}_{G_2}(-N_{G_2}) \otimes \mathcal{A}_2) \simeq \{\wedge^2 \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*\}(L_{\mathcal{Y}_2}) \simeq \rho_{\mathcal{Y}_2}^*(\mathcal{Q} \otimes \det \mathcal{Q}^*)(L_{\mathcal{Y}_2}).$$

By (4.8), we also have

$$\rho_{\mathcal{Y}_2}^*(\mathcal{Q} \otimes \det \mathcal{Q}^*)(L_{\mathcal{Y}_2}) \simeq \rho_{\mathcal{Y}_2}^* \mathcal{Q}(-M_{\mathcal{Y}_2} - F_\rho).$$

Therefore we have

$$\pi_{\mathcal{X}_2^t*}(\{(\bar{\rho}^t)^*\mathbf{S}^2\mathcal{W}\}(-N_{\mathcal{X}_2})) \simeq R^2 \mathrm{pr}_{2*}(\mathbf{S}^2\mathcal{W}_{G_2}(-N_{G_2}) \otimes \mathcal{A}_2) \simeq \rho_{\mathcal{Y}_2}^* \mathcal{Q}(-M_{\mathcal{Y}_2} - F_\rho)$$

as desired. \square

Step 4. Finally, in this step, we determine the sheaves in (6.19) and complete our construction of the locally free resolution of $\mathcal{I}_2^\circ \otimes \{\mathcal{O}_{\mathcal{Y}_2}(M_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}(2L_{\mathcal{X}})\}$ as follows:

$$\begin{aligned} 0 \rightarrow \rho_{\mathcal{Y}_2}^* \mathcal{S}^*(-L_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\mathcal{X}} &\rightarrow \mathcal{T}^* \boxtimes g^* \mathcal{F} \rightarrow \\ (6.22) \quad \mathcal{O}_{\mathcal{Y}_2} \boxtimes g^* \mathbf{S}^2 \mathcal{F} \oplus \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}(L_{\mathcal{X}}) &\rightarrow \\ \mathcal{I}_2^\circ \otimes \{\mathcal{O}_{\mathcal{Y}_2}(M_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\mathcal{X}}(2L_{\mathcal{X}})\} &\rightarrow 0. \end{aligned}$$

Our task is to show

Proposition 6.3.10. *It holds that*

- (1) $\pi_{\mathcal{X}_2*} \mathcal{O}_{\mathcal{X}_2}(N_{\mathcal{X}_2}) \simeq \rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$.
- (2) $\pi_{\mathcal{X}_2*}(\bar{\rho}^*\mathcal{W}) \simeq \mathcal{T}$ (cf. (6.1)), and
- (3) $\pi_{\mathcal{X}_2*}(\{\bar{\rho}^*\mathbf{S}^2\mathcal{W}\}(-N_{\mathcal{X}_2})) \simeq \rho_{\mathcal{Y}_2}^* \mathcal{Q}(-M_{\mathcal{Y}_2})$.

Proof. As we note after the statement of Lemma 6.3.9, we have a natural injection $\pi_{\mathcal{X}_2^t*}(\mathcal{B}|_{\mathcal{X}_2^t}) \hookrightarrow \pi_{\mathcal{X}_2*}(\mathcal{B}|_{\mathcal{X}_2})$ for any sheaf \mathcal{B} as in (6.19) which is isomorphic outside F_ρ . Let y be a point of F_ρ and q the fiber of $\mathcal{X}_2 \rightarrow \mathcal{Y}_2$ over y . Let $[V_1]$ be the image of y on $\mathbb{P}(V)$. We write $\mathbb{P}_q^2 = \mathbb{P}(W)$, where the plane \mathbb{P}_q^2 in $\mathbf{G}(2, V/V_1)$ is spanned by the conic q , and W is a three-dimensional subspace of $\wedge^2(V/V_1)$. Since q is a ρ -conic, there exists a 2-dimensional subspace V_2 such that $V_1 \subset V_2$ and $W = V/V_2 \otimes V_2/V_1 \simeq V/V_2$.

To see (1), we have only to show the injection

$$\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2}) \simeq \pi_{\mathcal{X}_2^t*} \mathcal{O}_{\mathcal{X}_2^t}(N_{\mathcal{X}_2^t}) \hookrightarrow \pi_{\mathcal{X}_2*} \mathcal{O}_{\mathcal{X}_2}(N_{\mathcal{X}_2})$$

is an isomorphism. This follows since both the fibers of $\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$ and $\pi_{\mathcal{X}_2*} \mathcal{O}_{\mathcal{X}_2}(N_{\mathcal{X}_2})$ at y are isomorphic to $H^0(\mathbb{P}_q^2, \mathcal{O}_{\mathbb{P}_q^2}(1))$.

We show (2). In the proof of Lemma 6.3.9 (ii), we show that $\mathrm{pr}_{2*}(\pi_{G_2}^*T(-1)) \simeq \pi_{\mathcal{X}_2^t*}((\bar{\rho}^t)^*\mathcal{W})$, where $\pi_{G_2}: G_2 \rightarrow \mathbb{P}(V)$ is the natural morphism. Note that \mathcal{W}_{G_2} is the pull-back of the universal quotient bundle on $\mathbf{G}(2, T(-1))$.

We compute the map

$$(6.23) \quad \mathrm{pr}_{2*}(\pi_{G_2}^*T(-1)) \otimes k(y) \simeq \pi_{\mathcal{X}_2^t*}((\bar{\rho}^t)^*\mathcal{W}) \otimes k(y) \rightarrow \pi_{\mathcal{X}_2*}(\bar{\rho}^*\mathcal{W}) \otimes k(y).$$

Since pr_{2*} and $\pi_{\mathcal{Z}_2}$ are flat near y , $\mathrm{pr}_{2*}(\pi_{G_2}^* T(-1)) \otimes k(y) \simeq H^0(G(2, V/V_1), V/V_1 \otimes \mathcal{O}_{G(2, V/V_1)}) \simeq V/V_1$ and $\pi_{\mathcal{Z}_2*}(\bar{\rho}^* \mathcal{W}) \otimes k(y) \simeq H^0(q, \bar{\rho}^* \mathcal{W}|_q)$ by the Grauert theorem. By Lemma 6.3.2, the map $V/V_1 \rightarrow H^0(q, \bar{\rho}^* \mathcal{W}|_q)$ factor through $H^0(\mathbb{P}_q^2, T_{\mathbb{P}_q^2}(-1)) \simeq V/V_2$. Therefore the cokernel of the dual of the map (6.23) is isomorphic to $(V_2/V_1)^*$, which can be identified with the fiber of $\rho_{\mathcal{Y}_2}^* \mathcal{O}_{\mathbb{P}(T(-1))}(1)$ at y . Hence it holds that $\{\pi_{\mathcal{Z}_2*}(\bar{\rho}^* \mathcal{W})\}^* \subset \mathcal{T}^*$ by (6.1). Since both the kernels of the maps

$$(\{\pi_{\mathcal{Z}_2*}(\bar{\rho}^* \mathcal{W})\}^*) \otimes k(y) \rightarrow (\{\pi_{\mathcal{Z}_2^t*}((\bar{\rho}^t)^* \mathcal{W})\}^*) \otimes k(y)$$

and

$$(\mathcal{T}^*) \otimes k(y) \rightarrow (\{\pi_{\mathcal{Z}_2^t*}((\bar{\rho}^t)^* \mathcal{W})\}^*) \otimes k(y)$$

are one-dimensional and the image of them coincide, actually we have $\{\pi_{\mathcal{Z}_2*}(\bar{\rho}^* \mathcal{W})\}^* = \mathcal{T}^*$ as desired.

Finally we show (3). Let $\bar{\mathcal{Z}}_\rho^u := \mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*|_{F_\rho})$ and $\mathcal{Z}_\rho := \mathcal{Z}_2|_{\bar{\mathcal{Z}}_\rho^u}$, which are the restrictions of \mathcal{Z}_2^u and \mathcal{Z}_2 over F_ρ . Note that $\bar{\mathcal{Z}}_\rho^u$ is the exceptional divisor of $\mathcal{Z}_2^u \rightarrow \mathcal{Z}_3^u$. We consider the diagram:

$$(6.24) \quad \begin{array}{ccc} \mathcal{Z}_\rho^u & \xleftarrow{\quad} & \bar{\mathcal{Z}}_\rho^u = \mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*|_{F_\rho}) \supset \mathcal{Z}_\rho \\ \pi_{\mathcal{Z}_\rho} \downarrow & & \downarrow \pi_{\bar{\mathcal{Z}}_\rho^u} \quad \swarrow \pi_{\mathcal{Z}_\rho} \\ \mathcal{P}_\rho & \xleftarrow{\quad \rho_{F_\rho} \quad} & F_\rho \end{array}$$

Let $\mathcal{W}_{\bar{\mathcal{Z}}_\rho^u}$ be the pull-back of $\mathcal{W}_{\mathcal{Z}_\rho^u}$ by $\bar{\mathcal{Z}}_\rho^u \rightarrow \mathcal{Z}_\rho^u$. By the proof of Lemma 6.3.8, we have

$$(6.25) \quad \pi_{\mathcal{Z}_2*}(\{\bar{\rho}^* \mathcal{S}^2 \mathcal{W}\}(-N_{\mathcal{Z}_2})|_{\mathcal{Z}_\rho}) \simeq R^1 \pi_{\bar{\mathcal{Z}}_\rho^u*}(\mathcal{S}^2 \mathcal{W}_{\bar{\mathcal{Z}}_\rho^u}(-N_{\mathcal{Z}_2^u} - \mathcal{Z}_2)|_{\bar{\mathcal{Z}}_\rho^u}).$$

It suffices to show that the r.h.s. is isomorphic to $\rho_{F_\rho}^* \mathcal{Q}(-M_{\mathcal{Y}_2}|_{F_\rho})$. In Lemma 6.3.11 below, we rewrite the r.h.s. by using the pull-back $\mathcal{R}_{\bar{\mathcal{Z}}_\rho^u}$ to $\bar{\mathcal{Z}}_\rho^u$ of $\mathcal{R}_{\mathcal{Z}_\rho^u}$ and divisors on F_ρ . Then, the calculation is reduced to that of $R^1 \pi_{\bar{\mathcal{Z}}_\rho^u*}(\mathcal{S}^2 \mathcal{R}_{\bar{\mathcal{Z}}_\rho^u}(-3 \det \mathcal{R}_{\bar{\mathcal{Z}}_\rho^u}))$, and, by applying the Bott theorem 2.1.1 to the projective bundle $\bar{\mathcal{Z}}_\rho^u = \mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*|_{F_\rho}) \rightarrow F_\rho$, the latter is isomorphic to

$$\rho_{F_\rho}^*(\mathcal{S} \otimes \det \mathcal{S}|_{\mathcal{P}_\rho}) \simeq \rho_{F_\rho}^*(\mathcal{Q} \otimes \det \mathcal{Q}|_{\mathcal{P}_\rho})(-4L_{F_\rho}),$$

where we use (4.6) for the second isomorphism. Therefore, for the r.h.s. of (6.25), we finally arrive at $\rho_{F_\rho}^* \mathcal{Q}(-M_{\mathcal{Y}_2}|_{F_\rho})$ by Lemma 6.3.11. \square

Lemma 6.3.11. *Let $\mathcal{R}_{\bar{\mathcal{Z}}_\rho^u}$ be the pull-back of $\mathcal{R}_{\mathcal{Z}_\rho^u}$. The r.h.s. of (6.25) is isomorphic to*

$$(6.26) \quad R^1 \pi_{\bar{\mathcal{Z}}_\rho^u*}(\mathcal{S}^2 \mathcal{R}_{\bar{\mathcal{Z}}_\rho^u}(-3 \det \mathcal{R}_{\bar{\mathcal{Z}}_\rho^u})) \otimes \mathcal{O}_{F_\rho}(-\rho_{F_\rho}^* \det \mathcal{Q}|_{\mathcal{P}_\rho} + 4L_{F_\rho} - M_{\mathcal{Y}_2}|_{F_\rho}).$$

Proof. First, we have

$$\mathcal{S}^2 \mathcal{W}_{\mathcal{Z}_\rho^u} \simeq \mathcal{S}^2 \mathcal{R}_{\mathcal{Z}_\rho^u} \otimes \pi_{\mathcal{Z}_\rho^u}^* \mathcal{O}_{\mathbb{P}(T(-1))}(2) \simeq \mathcal{S}^2 \mathcal{R}_{\mathcal{Z}_\rho^u}(\pi_{\mathcal{Z}_\rho^u}^*(\det \mathcal{Q}|_{\mathcal{P}_\rho}) - 2L_{\mathcal{Z}_\rho^u}),$$

where the first isomorphism follows from Lemma 5.2.7 and the second isomorphism follows from (4.7). Then, taking the pull-back to $\bar{\mathcal{Z}}_\rho^u$, we obtain

$$(6.27) \quad \mathcal{S}^2 \mathcal{W}_{\bar{\mathcal{Z}}_\rho^u} \simeq \mathcal{S}^2 \mathcal{R}_{\bar{\mathcal{Z}}_\rho^u}(\pi_{\bar{\mathcal{Z}}_\rho^u}^* \rho_{F_\rho}^*(\det \mathcal{Q}|_{\mathcal{P}_\rho}) - 2L_{\bar{\mathcal{Z}}_\rho^u}).$$

Second, by taking the determinants of (5.4), we have

$$(6.28) \quad H_{\mathbb{P}(\mathcal{S}^*|\mathcal{P}_\rho)} = \det \mathcal{R}_{\mathcal{P}_\rho^u} + \pi_{\mathcal{P}_\rho^u}^*(\det \mathcal{S}|\mathcal{P}_\rho) = \det \mathcal{R}_{\mathcal{P}_\rho^u} + \pi_{\mathcal{P}_\rho^u}^*(\det \mathcal{Q}|\mathcal{P}_\rho) - 3L_{\mathcal{P}_\rho^u},$$

where the second equality follows from (4.4). Therefore, by Proposition 5.2.4 (1), we obtain

$$N_{\mathcal{Z}_3^u|\mathcal{P}_\rho^u} = \det \mathcal{R}_{\mathcal{P}_\rho^u} + \pi_{\mathcal{P}_\rho^u}^*(\det \mathcal{Q}|\mathcal{P}_\rho) - 2L_{\mathcal{P}_\rho^u},$$

and then taking the pull-back to $\overline{\mathcal{Z}_\rho^u}$, we arrive at

$$(6.29) \quad N_{\mathcal{Z}_2^u|\overline{\mathcal{Z}_\rho^u}} = \det \mathcal{R}_{\overline{\mathcal{Z}_\rho^u}} + \pi_{\overline{\mathcal{Z}_\rho^u}}^*(\rho_{F_\rho}^* \det \mathcal{Q}|\mathcal{P}_\rho) - 2L_{\overline{\mathcal{Z}_\rho^u}}.$$

By (6.27) and (6.29), we have

$$(6.30) \quad S^2 \mathcal{W}_{\overline{\mathcal{Z}_\rho^u}}(-N_{\mathcal{Z}_2^u}|\overline{\mathcal{Z}_\rho^u}) \simeq S^2 \mathcal{R}_{\overline{\mathcal{Z}_\rho^u}}(-\det \mathcal{R}_{\overline{\mathcal{Z}_\rho^u}}).$$

Now we compute the class of the divisor $\mathcal{Z}_\rho = \mathcal{Z}_2|\overline{\mathcal{Z}_\rho^u}$ of $\overline{\mathcal{Z}_\rho^u}$. We see that $\mathcal{Z}_2 \in |2H_{\mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*)} + L_{\mathcal{Z}_2^u} - \pi_{\mathcal{Z}_2^u}^* F_\rho|$ since \mathcal{Z}_2 is the strict transform of \mathcal{Z}_3 by the blow-up $\mathcal{Z}_2^u \rightarrow \mathcal{Z}_3^u \simeq \mathbb{P}(\mathcal{S}^*)$ and $\mathcal{Z}_3 \in |2H_{\mathbb{P}(\mathcal{S}^*)} + L_{\mathcal{Z}_3^u}|$ by the proof of Proposition 5.2.4. By (4.8), we have

$$2H_{\mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*)} + L_{\mathcal{Z}_2^u} - \pi_{\mathcal{Z}_2^u}^* F_\rho = 2H_{\mathbb{P}(\rho_{\mathcal{Y}_2}^* \mathcal{S}^*)} + 2L_{\mathcal{Z}_2^u} - \pi_{\mathcal{Z}_2^u}^*(\rho_{F_\rho}^* \det \mathcal{Q}|\mathcal{P}_\rho) + \pi_{\mathcal{Z}_2^u}^*(M_{\mathcal{Y}_2}|_{F_\rho}).$$

Therefore, by (6.28), we obtain

$$(6.31) \quad \mathcal{Z}_\rho \in |2 \det \mathcal{R}_{\overline{\mathcal{Z}_\rho^u}} + \pi_{\overline{\mathcal{Z}_\rho^u}}^*(\rho_{F_\rho}^* \det \mathcal{Q}|\mathcal{P}_\rho) - 4L_{\overline{\mathcal{Z}_\rho^u}} + \pi_{\overline{\mathcal{Z}_\rho^u}}^*(M_{\mathcal{Y}_2}|_{F_\rho})|.$$

By (6.30) and (6.31), we finish the proof of the claim. \square

6.4. A locally free resolution of the ideal sheaf \mathcal{I} of Δ on $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$.

Set $\tilde{\rho}_{\mathcal{Y}_2}' := \tilde{\rho}_{\mathcal{Y}_2} \times \text{id}$. We calculate the pushforward $\tilde{\rho}_{\mathcal{Y}_2*}'$ of the exact sequence (6.22) on the locus $\mathcal{Y}_2^o \times \check{\mathcal{X}}$ to $\widetilde{\mathcal{Y}}^o \times \check{\mathcal{X}}$, where we set $\widetilde{\mathcal{Y}}^o := \widetilde{\mathcal{Y}} \setminus \mathcal{P}_\sigma$. To do this, we split (6.22) as follows:

$$(6.32) \quad 0 \rightarrow \rho_{\mathcal{Y}_2}^* \mathcal{S}^*(-L_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}} \rightarrow \mathcal{T}^* \boxtimes g^* \mathcal{F} \rightarrow \mathcal{C} \rightarrow 0$$

and

$$(6.33) \quad 0 \rightarrow \mathcal{C} \rightarrow \mathcal{O}_{\mathcal{Y}_2} \boxtimes g^* \mathcal{S}^2 \mathcal{F} \oplus \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(L_{\check{\mathcal{X}}}) \rightarrow \mathcal{I}_2^o \otimes \{\mathcal{O}_{\mathcal{Y}_2}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\} \rightarrow 0.$$

Since $\rho_{\mathcal{Y}_2}^* \mathcal{S}^*(-L_{\mathcal{Y}_2})$, \mathcal{T}^* and $\rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2})$ are the pull-backs of locally free sheaves $\tilde{\mathcal{S}}_L^*$, $\tilde{\mathcal{T}}^*$, and $\tilde{\mathcal{Q}}^*(M_{\widetilde{\mathcal{Y}}})$ on $\widetilde{\mathcal{Y}}$, the higher direct images of $\rho_{\mathcal{Y}_2}^* \mathcal{S}^*(-L_{\mathcal{Y}_2}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}$ and $\mathcal{T}^* \boxtimes g^* \mathcal{F}^*$ vanish. Therefore the pushforward of (6.32) is still exact and the higher direct images of \mathcal{C} vanish. Then the pushforward of (6.32) is also exact. Therefore we obtain the following exact sequence on $\widetilde{\mathcal{Y}}^o \times \check{\mathcal{X}}$:

$$(6.34) \quad 0 \rightarrow \tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}} \rightarrow \tilde{\mathcal{T}}^* \boxtimes g^* \mathcal{F} \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}} \boxtimes g^* \mathcal{S}^2 \mathcal{F} \oplus \tilde{\mathcal{Q}}^*(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(L_{\check{\mathcal{X}}}) \rightarrow \mathcal{I}^o \otimes \{\mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\} \rightarrow 0,$$

where $\mathcal{I}^o := \tilde{\rho}_{\mathcal{Y}_2*}' \mathcal{I}_2^o$.

We set $\tilde{\rho}_\Delta := \tilde{\rho}_{\mathcal{Y}_2}'|_{\Delta_2}$. To show Δ is normal and Cohen-Macaulay in Subsection 6.5, we prepare the following lemma:

Lemma 6.4.1. *Let Δ° be the closed subscheme of $\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}$ defined by \mathcal{I}° . Then $\tilde{\rho}_{\Delta^\circ}^* \mathcal{O}_{\Delta_2^\circ} = \mathcal{O}_{\Delta^\circ}$ and $R^1 \tilde{\rho}_{\Delta^\circ}^* \mathcal{O}_{\Delta_2^\circ} = 0$.*

Proof. By (6.32) and (6.33), we have $R^1 \tilde{\rho}_{\mathcal{Y}_2}^* \mathcal{I}_2^\circ = 0$ since $\rho_{\mathcal{Y}_2}^* \mathcal{S}^*(-L_{\mathcal{Y}_2})$, \mathcal{T}^* and $\rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2})$ are the pull-backs of locally free sheaves on $\widetilde{\mathcal{Y}}$. Taking the higher direct image of the exact sequence $0 \rightarrow \mathcal{I}_2^\circ \rightarrow \mathcal{O}_{\mathcal{Y}_2^\circ \times \check{\mathcal{X}}} \rightarrow \mathcal{O}_{\Delta_2^\circ} \rightarrow 0$, we obtain the exact sequence $0 \rightarrow \mathcal{I}^\circ \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}} \rightarrow \tilde{\rho}_{\Delta^\circ}^* \mathcal{O}_{\Delta_2^\circ} \rightarrow 0$ and $R^1 \tilde{\rho}_{\Delta^\circ}^* \mathcal{O}_{\Delta_2^\circ} = 0$ since $R^1 \tilde{\rho}_{\mathcal{Y}_2}^* \mathcal{I}_2^\circ = 0$ and $R^1 \tilde{\rho}_{\mathcal{Y}_2}^* \mathcal{O}_{\mathcal{Y}_2^\circ \times \check{\mathcal{X}}} = 0$. Thus $\tilde{\rho}_{\Delta^\circ}^* \mathcal{O}_{\Delta_2^\circ} = \mathcal{O}_{\Delta^\circ}$. \square

Let $\mathcal{I} := \iota_{\widetilde{\mathcal{Y}}^\circ}^* \mathcal{I}^\circ$, where we denote by $\iota_{\widetilde{\mathcal{Y}}^\circ}$ the embedding $\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$. We set $\Gamma_{\widetilde{\mathcal{Y}}} := \widetilde{\mathcal{Y}} \times \check{\mathcal{X}} \setminus \widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}$. Note that $\text{codim } \Gamma_{\widetilde{\mathcal{Y}}} = 6$ since the codimension of \mathcal{P}_σ in $\widetilde{\mathcal{Y}}$ is 6. Let

$$\mathcal{A} := \text{coker}(\tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}} \rightarrow \tilde{\mathcal{T}}^* \boxtimes g^* \mathcal{F})|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}.$$

We have the following exact sequences:

$$(6.35) \quad \begin{aligned} 0 \rightarrow \tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}} \rightarrow \tilde{\mathcal{T}}^* \boxtimes g^* \mathcal{F} \rightarrow \iota_{\widetilde{\mathcal{Y}}^\circ}^* \mathcal{A} \rightarrow R^1 \iota_{\widetilde{\mathcal{Y}}^\circ}^* (\tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) \\ \rightarrow R^1 \iota_{\widetilde{\mathcal{Y}}^\circ}^* (\tilde{\mathcal{T}}^* \boxtimes g^* \mathcal{F}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) \rightarrow R^1 \iota_{\widetilde{\mathcal{Y}}^\circ}^* \mathcal{A} \rightarrow R^2 \iota_{\widetilde{\mathcal{Y}}^\circ}^* (\tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}), \end{aligned}$$

and

$$(6.36) \quad \begin{aligned} 0 \rightarrow \iota_{\widetilde{\mathcal{Y}}^\circ}^* \mathcal{A} \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}} \boxtimes g^* \mathcal{S}^2 \mathcal{F} \oplus \tilde{\mathcal{Q}}^*(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(L_{\check{\mathcal{X}}}) \rightarrow \\ \mathcal{I}^\circ \otimes \{\mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\} \rightarrow R^1 \iota_{\widetilde{\mathcal{Y}}^\circ}^* \mathcal{A}, \end{aligned}$$

where we note that $\iota_{\widetilde{\mathcal{Y}}^\circ}^*(\mathcal{E}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) = \mathcal{E}$ for a locally free sheaf \mathcal{E} on $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$ since $\text{codim } \Gamma_{\widetilde{\mathcal{Y}}} \geq 2$, and $\iota_{\widetilde{\mathcal{Y}}^\circ}^*(\mathcal{I}^\circ \otimes \{\mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) = \mathcal{I} \otimes \{\mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\}$ by definition. For a sheaf \mathcal{E} on $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$, it holds that $R^i \iota_{\widetilde{\mathcal{Y}}^\circ}^*(\mathcal{E}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) = \mathcal{H}_{\Gamma_{\widetilde{\mathcal{Y}}}}^{i+1}(\mathcal{E})$ for $i > 0$ by [H, p.9, Corollary 1.9]. Moreover, if \mathcal{E} is locally free, then $\mathcal{H}_{\Gamma_{\widetilde{\mathcal{Y}}}}^{i+1}(\mathcal{E}) = 0$ for $i + 1 < 4$ by [H, p.44, Theorem 3.8] since $\text{codim } \Gamma_{\widetilde{\mathcal{Y}}} = 6$. Therefore,

$$R^1 \iota_{\widetilde{\mathcal{Y}}^\circ}^*(\tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) \simeq R^1 \iota_{\widetilde{\mathcal{Y}}^\circ}^*(\tilde{\mathcal{T}}^* \boxtimes g^* \mathcal{F}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) \simeq R^2 \iota_{\widetilde{\mathcal{Y}}^\circ}^*(\tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}}|_{\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}}) = 0.$$

Then, by (6.35), it holds that $R^1 \iota_{W*} \mathcal{A} = 0$. Consequently, the sequence (6.2) in Theorem 6.1.1 is exact on $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$ by (6.35) and (6.36).

6.5. Δ is normal and Cohen-Macaulay.

Now we show that Δ is normal and is Cohen-Macaulay, which completes the proof of Theorem 6.1.1.

We see that $\Delta_2^\mathcal{F}$ is smooth by Proposition 6.2.3, and $\Delta_2^\mathcal{F} \rightarrow \Delta_2$ is birational by Proposition 6.2.2 since $\Delta_2^\mathcal{F}$ and Δ_2 are birational to $\Delta_3^\mathcal{F}$ and Δ_3 respectively.

Recall that we set $\pi_\Delta := \tilde{\pi}_{\mathcal{Y}_2}|_{\Delta_2^\mathcal{F}}$ and $\tilde{\rho}_\Delta := \tilde{\rho}_{\mathcal{Y}_2}|_{\Delta_2^\mathcal{F}}$. We check the assertions separately on $\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}$ and $\mathcal{P}_\sigma \times \check{\mathcal{X}}$.

First we consider the problem on $\widetilde{\mathcal{Y}}^\circ \times \check{\mathcal{X}}$. For simplicity of notation, we do not use the symbols for the restrictions. We show that

$$(6.37) \quad R^i(\tilde{\rho}_\Delta \circ \pi_\Delta)_* \mathcal{O}_{\Delta_2^\mathcal{F}} = 0 \text{ for } i > 0$$

and

$$(6.38) \quad (\tilde{\rho}_\Delta \circ \pi_\Delta)_* \mathcal{O}_{\Delta_2^\mathcal{X}} \simeq \mathcal{O}_\Delta.$$

The latter shows that Δ is normal since $\Delta_2^\mathcal{X}$ is smooth, and the former show that Δ has only rational singularities, and then is Cohen-Macaulay. By using the Leray spectral sequence, (6.37) and (6.38) follow from similar statements for π_Δ and $\tilde{\rho}_\Delta$, which are Lemmas 6.3.5 and 6.4.1 respectively.

Second we consider the problem on $\mathcal{P}_\sigma \times \mathcal{X}$. By the first part, we see that Δ is regular in codimension one since the codimension of $\Delta \cap (\mathcal{P}_\sigma \times \mathcal{X})$ is greater than two. Therefore we have only to show that Δ is Cohen-Macaulay at any point of $\Delta \cap (\mathcal{P}_\sigma \times \mathcal{X})$. This follows from taking the local cohomology sequence of the locally free resolution (6.2) since $\widetilde{\mathcal{Y}} \times \mathcal{X}$ is smooth and the length of the locally free part of (6.2) is three.

Remark. Since we have shown Δ is reduced, Δ is the closure of Δ° .

7. THE UNIVERSAL FAMILY OF HYPERPLANE SECTIONS

Let $\mathcal{V} \subset \widetilde{\mathcal{Y}} \times \mathcal{X}$ be the pull-back of the universal family of hyperplane sections in $\mathbb{P}(\mathbf{S}^2 V^*) \times \mathbb{P}(\mathbf{S}^2 V)$. We can consider \mathcal{V} to be both the family of the pull-backs of hyperplanes of \mathcal{X} parameterized by $\widetilde{\mathcal{Y}}$, and the family of the pull-backs of hyperplanes of \mathcal{H} parameterized by \mathcal{X} . We simply say that \mathcal{V} is the family of hyperplane sections of $\widetilde{\mathcal{Y}}$ and \mathcal{X} . Note that the fiber $\subset \widetilde{\mathcal{Y}}$ of \mathcal{V} over a point $x \in X \subset \mathcal{X}$ is the pull-back of the hyperplane section $w_{xy}^\perp \cap \mathcal{H}$ of \mathcal{H} , where $x = w_{xy}$ as a point of $\mathbf{S}^2 \mathbb{P}(V)$.

In this section, we show two results on \mathcal{V} ; In Proposition 7.0.1, we show that Δ is a closed subscheme of \mathcal{V} . This is technically important to show $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ (see Lemma 9.0.6), and should be theoretically essential to show that (suitable noncommutative resolutions of) \mathcal{Y} and \mathcal{X} are homologically projective dual to each other (cf. [Ku1, Def.6.1]). Moreover, in Proposition 7.0.1, we give a locally free resolution of the ideal sheaf of Δ in \mathcal{V} as an $\mathcal{O}_{\widetilde{\mathcal{Y}} \times \mathcal{X}}$ -module. It should be observed that the locally free sheaves in this resolution comprise the (dual) Lefschetz collections in $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ and $\mathcal{D}^b(\mathcal{X})$ which are obtained in [HoTa3]. In Proposition 7.0.2, we show that any hyperplane section of $\widetilde{\mathcal{Y}}$ corresponding to a point of \mathcal{X} has only canonical singularities. This is also technically important to apply the Kawamata-Viehweg vanishing theorem in the proof of the derived equivalence (see the proof of Claim 9.0.8).

We begin with a preliminary discussion. The ideal sheaf $\mathcal{I}_\mathcal{V}$ of \mathcal{V} on $\widetilde{\mathcal{Y}} \times \mathcal{X}$ is isomorphic to $\mathcal{O}_{\widetilde{\mathcal{Y}}}(-M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\mathcal{X}}(-H_{\mathcal{X}})$. Note that \mathcal{V} has a natural $\mathrm{SL}(V)$ -action. Therefore the injection $\mathcal{O}_{\widetilde{\mathcal{Y}}}(-M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\mathcal{X}}(-H_{\mathcal{X}}) \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}} \times \mathcal{X}}$ is $\mathrm{SL}(V)$ -equivariant. We can apply the construction as in Subsection 6.1 to this map since $\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{Y}}}(-M_{\widetilde{\mathcal{Y}}}), \mathcal{O}_{\widetilde{\mathcal{Y}}}) \simeq \mathbf{S}^2 V$ and $\mathrm{Hom}(\mathcal{O}_{\mathcal{X}}(-H_{\mathcal{X}}), \mathcal{O}_{\mathcal{X}}) \simeq \mathbf{S}^2 V^*$. Since the above injection is $\mathrm{SL}(V)$ -equivariant, and $\mathbf{S}^2 V \otimes \mathbf{S}^2 V^*$ contains a unique one-dimensional representation, which is generated by the identity element, we see that the above injection is induced from the identity element as in Subsection 6.1.

Proposition 7.0.1. \mathcal{I} contains $\mathcal{I}_\mathcal{V}$, equivalently, the subvariety Δ is contained in \mathcal{V} . Set $\mathcal{I}_{\Delta/\mathcal{V}} := \mathcal{I}/\mathcal{I}_\mathcal{V}$, the ideal sheaf of Δ in \mathcal{V} . Denote by $\iota_\mathcal{V}$ the closed immersion $\mathcal{V} \hookrightarrow \widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$. Then $\iota_{\mathcal{V}*}\mathcal{I}_{\Delta/\mathcal{V}}$ has the following locally free resolution on $\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}$:

$$(7.1) \quad \begin{aligned} 0 \rightarrow \tilde{\mathcal{S}}_L^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}} &\rightarrow \tilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F} \rightarrow \\ \mathcal{O}_{\widetilde{\mathcal{Y}}} \boxtimes \{g^*\mathcal{S}^2\mathcal{F}/\mathcal{O}_{\check{\mathcal{X}}}(-H_{\check{\mathcal{X}}} + 2L_{\check{\mathcal{X}}})\} \oplus \tilde{\mathcal{Q}}^*(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(L_{\check{\mathcal{X}}}) &\rightarrow \\ \iota_{\mathcal{V}*}\mathcal{I}_{\Delta/\mathcal{V}} \otimes \{\mathcal{O}_{\widetilde{\mathcal{Y}}}(M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})\} &\rightarrow 0. \end{aligned}$$

where the inclusion $\mathcal{O}_{\check{\mathcal{X}}}(-H_{\check{\mathcal{X}}} + 2L_{\check{\mathcal{X}}}) \subset g^*\mathcal{S}^2\mathcal{F}$ is obtained from a part of the Euler sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{S}^2\mathcal{F}^*)}(-1) \rightarrow g^*\mathcal{S}^2\mathcal{F}^* \rightarrow T_{\mathbb{P}(\mathcal{S}^2\mathcal{F}^*)/\mathbb{G}(2,V)}(-1) \rightarrow 0$ for $\check{\mathcal{X}} = \mathbb{P}(\mathcal{S}^2\mathcal{F}^*)$ by tensoring $\mathcal{O}_{\check{\mathcal{X}}}(2L_{\check{\mathcal{X}}})$.

Proof. We have an $\mathrm{SL}(V)$ -equivariant map

$$\mathcal{O}_{\widetilde{\mathcal{Y}}}(-M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(-H_{\check{\mathcal{X}}}) \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}}(-M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{S}^2(g^*\mathcal{F}^*),$$

which is induced from the dual of the natural surjection $\mathcal{S}^2(g^*\mathcal{F}) \rightarrow \mathcal{O}_{\check{\mathcal{X}}}(H_{\check{\mathcal{X}}}) = \mathcal{O}_{\mathbb{P}(\mathcal{S}^2\mathcal{F}^*)}(1)$. Therefore we have a $\mathrm{SL}(V)$ -equivariant map

$$\mathcal{O}_{\widetilde{\mathcal{Y}}}(-M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(-H_{\check{\mathcal{X}}}) \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}}(-M_{\widetilde{\mathcal{Y}}}) \boxtimes \mathcal{S}^2(g^*\mathcal{F}^*) \oplus \tilde{\mathcal{Q}}^* \boxtimes \mathcal{O}_{\check{\mathcal{X}}}(-L_{\check{\mathcal{X}}}) \rightarrow \mathcal{I} \hookrightarrow \mathcal{O}_{\widetilde{\mathcal{Y}} \times \check{\mathcal{X}}}.$$

By the uniqueness of such a map, its image coincides with $\mathcal{I}_\mathcal{V}$. Therefore $\mathcal{I}_\mathcal{V} \subset \mathcal{I}$.

The proof of the latter assertion follows from the above discussion. \square

Now we give a description of singularities of fibers of $\mathcal{V} \rightarrow \check{\mathcal{X}}$.

Recall that $f: \check{\mathcal{X}} \rightarrow \mathcal{X}$ is the Hilbert-Chow morphism and E_f is the f -exceptional divisor as in Subsection 2.2. Let x be a point of $\mathbb{P}(V)$ and e any point of E_f such $f(e) = [2x]$. Then the fiber of $\mathcal{V} \rightarrow \check{\mathcal{X}}$ over e is the pull-back of the hyperplane section of \mathcal{H} parameterizing singular quadrics which contain the point x by the duality between $\mathbb{P}(\mathcal{S}^2V)$ and $\mathbb{P}(\mathcal{S}^2V^*)$. In particular, the fiber is independent of a choice of e once we fix a point x , thus we denote it by V_x .

Proposition 7.0.2. *Any fiber of $\mathcal{V} \rightarrow \check{\mathcal{X}}$ is normal and has only canonical singularities.*

Proof. The proof is similar to the argument in Subsection 6.5.

It suffices to show the assertion for V_x ($x \in \mathbb{P}(V)$) since it is a special fiber of $\mathcal{V} \rightarrow \check{\mathcal{X}}$. Let V_x^t is the strict transform of V_x on \mathcal{B}_2 , which is also the total transform since V_x does not contain the center $G_{\widetilde{\mathcal{Y}}}$ of the birational morphism $\mathcal{B}_2 \rightarrow \widetilde{\mathcal{Y}}$. Hence $V_x^t \in |M_{\mathcal{B}_2}|$. We see that $-K_{V_x^t}$ is $(\tilde{\rho}_{\mathcal{B}_2}|_{V_x^t})$ -ample since $V_x^t \in |M_{\mathcal{B}_2}|$ and $-K_{\mathcal{B}_2} = 10M_{\mathcal{B}_2}$ is $\tilde{\rho}_{\mathcal{B}_2}$ -ample. Therefore it suffices to show similar assertions for V_x^t . Let W_G , W_3 and W_2 be the pull-backs on $\mathbb{G}(2, T(-1))$, \mathcal{L}_3 , and \mathcal{L}_2 , respectively, of the subvariety $W_0 := \{\Pi \mid x \in \Pi\} \simeq \mathbb{G}(2, 4)$ in $\mathbb{G}(3, V)$. It is easy to see that V_x^t is the image of W_2 set-theoretically and $W_2 \rightarrow V_x^t$ is birational (note that, once we fix a general quadric Q and a \mathbb{P}^1 -family q of planes in Q , there is only one plane in q which contains x). We show that W_2 is smooth, therefore $W_2 \rightarrow V_x^t$ is a resolution of singularities. Indeed, W_3 is smooth since $W_0 \simeq \mathbb{G}(2, 4)$, $W_G \rightarrow W_0$ is a \mathbb{P}^2 -bundle, and $W_3 \rightarrow W_G$ is a $\mathbb{G}(2, 5)$ -bundle by Proposition 5.2.5. Then W_2

is also smooth since $W_2 \rightarrow W_3$ is the blow-up along $\pi_{\mathcal{Z}_3}^{-1}(\mathcal{P}_\rho) \cap W_3$, which is a \mathbb{P}^1 -bundle over W_G by Proposition 5.2.6, and hence is smooth.

Similarly to the proof of Proposition 6.3.1, we see that the ideal sheaf \mathcal{I}_{Γ_x} of Γ_x has the following Koszul resolution:

$$(7.2) \quad 0 \rightarrow \wedge^2 \mathcal{W}^* \rightarrow \mathcal{W}^* \rightarrow \mathcal{I}_{\Gamma_x} \rightarrow 0.$$

First we check the assertions on \mathcal{Y}_2^o . For simplicity of notation, we abbreviate the symbols for the restrictions. Since V_x^t is Gorenstein, we have only to show V_x^t is normal and has only rational singularities. In the same way to show (6.15), we obtain $\pi_{\mathcal{Z}_2*} \bar{\rho}^* \mathcal{W}^* = R^1 \pi_{\mathcal{Z}_2*} \bar{\rho}^* \mathcal{W}^* = 0$. Therefore we have

$$(7.3) \quad \pi_{\mathcal{Z}_2*} \mathcal{I}_{W_2} \simeq R^1 \pi_{\mathcal{Z}_2*} \wedge^2 \bar{\rho}^* \mathcal{W}^*,$$

where \mathcal{I}_{W_2} is the ideal sheaf of W_2 in \mathcal{Y}_2 . In the same way to show (6.17), we obtain by the Grothendieck-Verdier duality 2.1.2

$$R^1 \pi_{\mathcal{Z}_2*} \wedge^2 \bar{\rho}^* \mathcal{W}^* \simeq (\pi_{\mathcal{Z}_2*} \{ \wedge^2 \bar{\rho}^* \mathcal{W} \otimes \mathcal{O}_{\mathcal{Z}_2}(-N_{\mathcal{Z}_2}) \} \otimes \mathcal{O}_{\mathcal{Y}_2}(M_{\mathcal{Y}_2}))^* \simeq \mathcal{O}_{\mathcal{Y}_2}(-M_{\mathcal{Y}_2}).$$

Therefore $\pi_{\mathcal{Z}_2*} \mathcal{I}_{W_2}$ is nothing but the ideal sheaf $\mathcal{I}_{V_x^t}$ of V_x^t since V_x^t is the image of W_2 set-theoretically. In other words, V_x^t is the scheme-theoretic pushforward of W_2 . We set $\pi_W := \pi_{\mathcal{Z}_2}|_{W_2} : W_2 \rightarrow V_x^t$. In the same way to show Lemma 6.4.1, we also have

$$(7.4) \quad R^k \pi_{W*} \mathcal{O}_{W_2} = 0 \text{ for } k > 0 \text{ and } \pi_{W*} \mathcal{O}_{W_2} \simeq \mathcal{O}_{V_x^t}.$$

Since W_2 is smooth, the latter shows that V_x^t is normal and the former shows that V_x^t has only rational singularities.

Second we show the assertions on the whole \mathcal{Y}_2 . By the above argument on \mathcal{Y}_2^o , we see that V_x^t is regular in codimension one since the codimension of $V_x^t \cap \mathcal{P}_\sigma$ in V_x^t is greater than two. Therefore V_x^t is normal since V_x^t is Gorenstein. To check V_x^t has only canonical singularities, we have only to show that $W_2 \rightarrow V_x^t$ is crepant since W_2 is smooth. This follows by calculating the canonical divisor of W_2 . Note that $\mathcal{N}_{W_2/\mathcal{Z}_2} \simeq \bar{\rho}^* \mathcal{W}^*|_{W_2}$ by (7.2). Therefore $\det \mathcal{N}_{W_2/\mathcal{Z}_2} \simeq \mathcal{O}_{\mathcal{Z}_2}(-N_{\mathcal{Z}_2})|_{W_2}$ since $\det \mathcal{W} = \mathcal{O}_{G(3,V)}(1)$. Thus we have

$$\begin{aligned} K_{W_2} &= K_{\mathcal{Z}_2}|_{W_2} + \det \mathcal{N}_{W_2/\mathcal{Z}_2} \\ &= \{ \pi_{\mathcal{Z}_2}^* (K_{\mathcal{Z}_2} + M_{\mathcal{Z}_2}) - N_{\mathcal{Z}_2} \}|_{W_2} + N_{\mathcal{Z}_2}|_{W_2} = \pi_{\mathcal{Z}_2}^* (K_{\mathcal{Z}_2} + M_{\mathcal{Z}_2})|_{W_2}, \end{aligned}$$

where the second equality follows from Proposition 5.3.1. □

8. THE FAMILY OF CURVES ON Y PARAMETERIZED BY X REVISITED

Let X and Y be smooth Calabi-Yau threefolds which are mutually orthogonal linear sections of \mathcal{X} and \mathcal{Y} respectively, as described in Subsection 2.4 and Section 3. In this section, we show that a family of curves on Y parameterized by X comes out from $\Delta \rightarrow \mathcal{X}$ and this family coincides with one constructed in Section 3. We prove also the family is flat.

8.1. Flatness of $\Delta \rightarrow \check{\mathcal{X}}$ and the locally free resolution of \mathcal{I}_x .

As applications of the descriptions of Δ in Section 6, we show that $\Delta \rightarrow \check{\mathcal{X}}$ is flat, and obtain locally free resolutions of the ideal sheaves of its fibers. Flatness of $\Delta \rightarrow \check{\mathcal{X}}$ will imply that of the family of curves on Y cut out from Δ in the next subsection. The locally free resolutions of its fibers will play an important role in the proof of the derived equivalence of X and Y (see Proposition 9.0.5).

We denote by (-1) the tensor product of $\mathcal{O}_{\check{\mathcal{Y}}}(-M_{\check{\mathcal{Y}}})$.

Proposition 8.1.1. 1) *The scheme Δ is flat over $\check{\mathcal{X}}$.* 2) *Let Δ_x be the fiber of $\Delta \rightarrow \check{\mathcal{X}}$ over a point $x \in \check{\mathcal{X}}$. Then the ideal sheaf \mathcal{I}_x of Δ_x in $\check{\mathcal{Y}}$ is $\mathcal{I} \otimes_{\mathcal{O}_{\check{\mathcal{Y}} \times \check{\mathcal{X}}}} \mathcal{O}_{\check{\mathcal{Y}}_x}$, where $\check{\mathcal{Y}}_x \simeq \check{\mathcal{Y}}$ is the fiber of $\check{\mathcal{Y}} \times \check{\mathcal{X}} \rightarrow \check{\mathcal{X}}$ over x . Moreover, the exact sequence (6.2) remains to be exact after restricting on $\check{\mathcal{Y}}_x$ and gives the following locally free resolution of \mathcal{I}_x :*

$$(8.1) \quad 0 \rightarrow \tilde{\mathcal{S}}_L^*(-1) \rightarrow \tilde{\mathcal{T}}^*(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\check{\mathcal{Y}}}(-1)^{\oplus 3} \oplus \tilde{\mathcal{Q}}^* \rightarrow \mathcal{I}_x \rightarrow 0.$$

Proof. By Proposition 6.2.2, $\Delta \rightarrow \check{\mathcal{X}}$ is equi-dimensional (actually all fibers are isomorphic). Therefore Δ is flat over $\check{\mathcal{X}}$ since Δ is Cohen-Macaulay and $\check{\mathcal{X}}$ is smooth.

Tensoring the exact sequence $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\check{\mathcal{Y}} \times \check{\mathcal{X}}} \rightarrow \mathcal{O}_{\Delta} \rightarrow 0$ with $k(x)$, we obtain the exact sequence $\mathcal{I} \otimes k(x) \rightarrow \mathcal{O}_{\check{\mathcal{Y}}_x} \rightarrow \mathcal{O}_{\Delta_x} \rightarrow 0$. Then, applying [M, Theorem 22.5 (1) \Rightarrow (2)], we see that $\mathcal{I} \otimes k(x) \rightarrow \mathcal{O}_{\check{\mathcal{Y}}_x}$ is injective since Δ is flat over $\check{\mathcal{X}}$. Note that, for any coherent $\mathcal{O}_{\check{\mathcal{Y}} \times \check{\mathcal{X}}}$ -module \mathcal{A} and a point $x \in \check{\mathcal{X}}$, it holds that $\mathcal{A} \otimes_{\mathcal{O}_{\check{\mathcal{Y}} \times \check{\mathcal{X}}}} \mathcal{O}_{\check{\mathcal{Y}}_x} \simeq \mathcal{A} \otimes_{\mathcal{O}_{\check{\mathcal{X}}}} k(x)$ since $\mathcal{O}_{\check{\mathcal{Y}}_x} \simeq \mathcal{O}_{\check{\mathcal{Y}} \times \check{\mathcal{X}}} \otimes_{\mathcal{O}_{\check{\mathcal{X}}}} k(x)$. Therefore $\mathcal{I}_x \simeq \mathcal{I} \otimes k(x) \simeq \mathcal{I} \otimes_{\mathcal{O}_{\check{\mathcal{Y}} \times \check{\mathcal{X}}}} \mathcal{O}_{\check{\mathcal{Y}}_x}$.

Proof of the exactness of (8.1) is similar. \square

8.2. Cutting a family of curves $\mathcal{C} \rightarrow X$ from Δ .

As we noted in Subsection 2.4, we may consider $X \subset \check{\mathcal{X}}$ and Y is disjoint from $G_{\mathcal{Y}} = \text{Sing } \mathcal{Y}$. Since $\check{\mathcal{Y}} \rightarrow \mathcal{Y}$ is isomorphism outside $G_{\mathcal{Y}}$, we may consider $Y \subset \check{\mathcal{Y}}$. Note that the subvariety $Y \times X$ of $\check{\mathcal{Y}} \times \check{\mathcal{X}}$ is contained in \mathcal{V} since X and Y are mutually orthogonal. Now we set

$$\mathcal{C} = \Delta|_{X \times Y}.$$

Let I be the ideal sheaf of \mathcal{C} in $Y \times X$ and I_x the ideal sheaf of C_x in Y_x , where $Y_x \simeq Y$ is the fiber of $Y \times X \rightarrow X$ over x .

Proposition 8.2.1. *The scheme \mathcal{C} is flat over X and its fiber over $x \in X$ coincides with C_x defined in Subsection 3.1. Moreover, it holds that $I \simeq \mathcal{I}_{\Delta/\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{Y \times X}$ and $I_x \simeq \mathcal{I}_{\Delta_x/\mathcal{Y}_x} \otimes_{\mathcal{O}_{\check{\mathcal{Y}}_x}} \mathcal{O}_{Y_x}$.*

Proof. Recall that $\pi_{\mathcal{Z}}: \mathcal{Z} \rightarrow \mathcal{Y}$ as in Subsection 2.3 coincides outside $G_{\mathcal{Y}}$ with the universal family of conics $\pi_{\mathcal{Z}_0}: \mathcal{Z}_0 \rightarrow \mathcal{Y}_0$ and parameterizes smooth τ - and ρ -conics (see Subsection 5.1). Let $Z := \pi_{\mathcal{Z}}^{-1}(Y)$.

Now we identify the generically conic bundles $\mathcal{Z} \rightarrow \mathcal{Y}$ and $\check{\mathcal{Z}} \rightarrow \check{\mathcal{Y}}$ (see the end of Subsection 5.3) near Y since the latter also coincides with the universal family of conics $\mathcal{Z}_0 \rightarrow \mathcal{Y}_0$ near Y .

Recall the commutative diagram (5.5). Let $\Delta^{\mathcal{Z}} \subset \widetilde{\mathcal{Z}}$ be the image of $\Delta_2^{\mathcal{Z}}$ on $\widetilde{\mathcal{Z}} \times \mathcal{X}$. Let $\Delta_x^{\mathcal{Z}}$ be the fiber of $\Delta^{\mathcal{Z}} \rightarrow \mathcal{X}$ over $x \in X$. Then we may identify γ_x as in Subsection 3.1 and $\Delta_x^{\mathcal{Z}} \cap Z$ since G_x as in loc.cit. is the fiber of $\Delta_0 \rightarrow \mathbf{G}(2, V)$ over $[l_x]$. By the commutativity of the diagram (5.5), we see that $\Delta \subset \mathcal{Y} \times \mathcal{X}$ coincides with the image of $\Delta^{\mathcal{Z}}$ outside $\mathcal{P}_\sigma \times \mathcal{X}$. Therefore, C_x , which is the image on Y of γ_x coincides with $\Delta_x \cap Y$, which is the fiber of $\mathcal{C} \rightarrow X$. Hereafter we denote by C_x the fiber of $\mathcal{C} \rightarrow X$ over $x \in X$.

Recall that the fiber \mathcal{V}_x of $\mathcal{V} \rightarrow \mathcal{X}$ over x contains Δ_x by Proposition 7.0.1. Since \mathcal{V}_x contains also Y , we see that Y is the complete intersection in \mathcal{V}_x of 9 members M_1, \dots, M_9 of $|M_{\widetilde{\mathcal{Y}}}|_{\mathcal{V}_x}|$. Therefore C_x is cut out from Δ_x in \mathcal{V}_x by M_1, \dots, M_9 . By [M, Corollary to Theorem 23.3], Δ_x is Cohen-Macaulay for any $x \in \mathcal{X}$ since Δ is Cohen-Macaulay by Theorem 6.1.1 and is flat over \mathcal{X} by Proposition 8.1.1. Since C_x is one-dimensional for any $x \in X$, the divisors M_1, \dots, M_9 form a regular sequence [M, Theorem 17.4 iii)]. Note that $\Delta|_{\widetilde{\mathcal{Y}} \times X} \rightarrow X$ is flat by Proposition 8.1.1 and its ideal sheaf is $\mathcal{I} \otimes \mathcal{O}_{\widetilde{\mathcal{Y}} \times X}$ by [M, Theorem 22.5 (2) \Rightarrow (1)]. Therefore, by [M, Corollary to Theorem 22.5 (2) \Rightarrow (1)], \mathcal{C} is flat over X and is cut out in \mathcal{V} from $\Delta|_{\widetilde{\mathcal{Y}} \times X}$ by a regular sequence. The latter implies that $\mathcal{I}_{\Delta/\mathcal{V}} \otimes_{\mathcal{O}_{\widetilde{\mathcal{Y}} \times \mathcal{X}}} \mathcal{O}_{Y \times X}$ is the ideal sheaf of \mathcal{C} in $X \times Y$. Similarly, we have $I_x = \mathcal{I}_{\Delta_x/\mathcal{V}_x} \otimes_{\mathcal{O}_{\widetilde{\mathcal{Y}}_x}} \mathcal{O}_{Y_x}$ since the scheme C_x is cut out in \mathcal{V}_x from Δ_x by a regular sequence. \square

9. DERIVED EQUIVALENCE

In this section, we derive the main result of this article:

Theorem 9.0.2. *Let X and Y be smooth Calabi-Yau threefolds which are mutually orthogonal linear sections of \mathcal{X} and \mathcal{Y} respectively. Let I be the ideal sheaf as in Proposition 8.2.1. Then the Fourier-Mukai functor Φ_I with I as its kernel is an equivalence between $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$.*

Since the cohomology groups related to the locally free resolution of \mathcal{I} have been computed in [HoTa3, Thm.8.1.1], the rest of our proof of the derived equivalence between X and Y proceeds in the same way as that of [BC].

We show that the functor $\Phi_I : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ is an equivalence by verifying the conditions (i) and (ii) of Theorem 2.1.3. The condition (i) may be verified by the following general lemma [BC, Proposition 4.5]. We include the proof for completeness.

Lemma 9.0.3. *Let Y be a smooth projective threefold and I an ideal sheaf of \mathcal{O}_Y such that the closed subscheme C defined by I is of (not necessarily pure) dimension less than or equal to one. Then $\mathrm{Hom}(I, I) \simeq \mathbb{C}$.*

Proof. Taking $\mathrm{Hom}(I, -)$ of the exact sequence

$$(9.1) \quad 0 \rightarrow I \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_C \rightarrow 0,$$

we obtain an injection $\mathrm{Hom}(I, I) \rightarrow \mathrm{Hom}(I, \mathcal{O}_Y)$. We have only to show $\mathrm{Hom}(I, \mathcal{O}_Y) \simeq \mathbb{C}$ since $\mathrm{Hom}(I, I)$ contains at least constant maps. To compute $\mathrm{Hom}(I, \mathcal{O}_Y)$, we take $\mathrm{Hom}(-, \mathcal{O}_Y)$ of (9.1). Then we obtain the exact sequence $0 \rightarrow \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) \rightarrow \mathrm{Hom}(I, \mathcal{O}_Y) \rightarrow \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_Y)$. By the Serre duality, we have $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_Y) \simeq$

$H^2(Y, \mathcal{O}_C \otimes \omega_Y)$, where the r.h.s. is 0 since $\dim C \leq 1$. Therefore we have $\text{Hom}(I, \mathcal{O}_Y) \rightarrow \text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \mathbb{C}$. \square

In what follows, we show the property (ii) of Theorem 2.1.3, i.e., the vanishing:

$$(9.2) \quad \text{Ext}^\bullet(I_{x_1}, I_{x_2}) = 0 \text{ for any two distinct points } x_1 \text{ and } x_2 \text{ of } X.$$

We denote by $(-t)$ the tensor product of $\mathcal{O}_{\tilde{\mathcal{Y}}}(-tM_{\tilde{\mathcal{Y}}})$. Then the result (3) of [HoTa3, Thm.8.1.1] may be read as:

Theorem 9.0.4. $H^\bullet(\mathcal{A}^* \otimes \mathcal{B}(-t)) = 0$ ($1 \leq t \leq 9$), where \mathcal{A} and \mathcal{B} , respectively, are one of the sheaves $\tilde{\mathcal{S}}_L^*$, $\tilde{\mathcal{T}}^*$, $\mathcal{O}_{\tilde{\mathcal{Y}}}$, $\tilde{\mathcal{Q}}^*(M_{\tilde{\mathcal{Y}}})$.

It is standard to derive the following proposition from Proposition 8.1.1 and Theorem 9.0.4.

Proposition 9.0.5. *For any two points x_1 and x_2 of $\tilde{\mathcal{X}}$, it holds*

$$\text{Ext}^\bullet(\mathcal{I}_{x_1}, \mathcal{I}_{x_2}(-t)) = 0 \quad (1 \leq t \leq 9).$$

We derive the vanishing (9.2) from Proposition 9.0.5 following [BC, Subsection 5.6]. For the derivation, it is important to choose a suitable sequence of the complete intersections of the members of $|M_{\tilde{\mathcal{Y}}}|$ containing Y .

Lemma 9.0.6. *There exists a tower of complete intersections of $\tilde{\mathcal{Y}}$ by members of $|M_{\tilde{\mathcal{Y}}}|$;*

$$Y_0 \subset Y_1 \subset \cdots \subset Y_9 \subset Y_{10} \quad (\dim Y_i = 3 + i)$$

satisfying the following conditions, where we set $\Delta_{x_i;j} := \Delta_{x_i}|_{Y_j}$ and denote by $\mathcal{I}_{x_i;j}$ the ideal sheaf of $\Delta_{x_i;j}$ in Y_j and by ι_{Y_j} the embedding $Y_j \hookrightarrow Y_{j+1}$:

- (1) $Y_0 = Y$ and $Y_{10} = \tilde{\mathcal{Y}}$.
- (2) $Y_9 = \mathcal{V}_{x_1}$, where \mathcal{V}_{x_1} is the fiber of $\mathcal{V} \rightarrow \tilde{\mathcal{X}}$ over x_1 (cf. Section 7). In particular, Y_9 contains Δ_{x_1} (Proposition 7.0.1), and hence $\iota_{Y_9}^* \mathcal{I}_{x_1;9} = \mathcal{I}_{x_1}/\mathcal{O}_{\tilde{\mathcal{Y}}}(-Y_9) \simeq \mathcal{I}_{x_1}/\mathcal{O}_{\tilde{\mathcal{Y}}}(-1)$. The ideal sheaf $\mathcal{I}_{x_2;9}$ is equal to $\mathcal{I}_{x_2} \otimes \mathcal{O}_{Y_9} \simeq \iota_{Y_9}^* \mathcal{I}_{x_2}$.
- (3) $Y_8 = \mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$, where the intersection is taken in $\tilde{\mathcal{Y}}$. In particular, Y_8 contains $\Delta_{x_2}|_{Y_9}$, and hence $\iota_{Y_8}^* \mathcal{I}_{x_2;8} = \mathcal{I}_{x_2;9}/\mathcal{O}_{Y_9}(-Y_8) \simeq \mathcal{I}_{x_2;9}/\mathcal{O}_{Y_9}(-1)$. The ideal sheaf $\mathcal{I}_{x_1;8}$ is equal to $\mathcal{I}_{x_1;9} \otimes \mathcal{O}_{Y_8} \simeq \iota_{Y_8}^* \mathcal{I}_{x_1;9}$.
- (4) For any $j \leq 7$, the ideal sheaf $\mathcal{I}_{x_i;j}$ is equal to $\mathcal{I}_{x_i;j+1} \otimes \mathcal{O}_{Y_j} \simeq \iota_{Y_j}^* \mathcal{I}_{x_i;j+1}$.

Proof. We take $Y_9 = \mathcal{V}_{x_1}$ and $Y_8 = \mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$ as in the statement and let Y_7, \dots, Y_0 be general complete intersections containing Y (recall that Y is contained in any fiber of $\mathcal{V} \rightarrow \tilde{\mathcal{X}}$ over a point of X). Note that \mathcal{V}_{x_1} is irreducible by Proposition 7.0.2, and $\mathcal{V}_{x_1} \neq \mathcal{V}_{x_2}$ by the duality of $\mathbb{P}(\mathbb{S}^2 V)$ and $\mathbb{P}(\mathbb{S}^2 V^*)$ since $x_1 \neq x_2$ and we may consider X is embedded in $\tilde{\mathcal{X}} \subset \mathbb{P}(\mathbb{S}^2 V)$.

The descriptions of $\mathcal{I}_{x_2;9}$, $\mathcal{I}_{x_1;8}$ and $\mathcal{I}_{x_i;j}$ ($i = 1, 2, 0 \leq j \leq 7$) follow from the last part of the proof of Proposition 8.2.1. \square

The choices of Y_9 and Y_8 in the lemma turns out to be crucial in Steps 1 and 2 of the arguments below.

Step 1 (from $\widetilde{\mathcal{Y}}$ to Y_9).

In this step, we show

$$(9.3) \quad \mathrm{Ext}_{Y_9}^{\bullet-1}(\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2;9}(-t+1)) = 0 \quad (1 \leq t \leq 9).$$

By $\mathcal{I}_{x_2;9} \simeq \iota_{Y_9}^* \mathcal{I}_{x_2}$, we have

$$(9.4) \quad \mathrm{Ext}_{Y_9}^{\bullet-1}(\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2;9}(-t+1)) \simeq \mathrm{Ext}_{Y_9}^{\bullet-1}(\mathcal{I}_{x_1;9}, \iota_{Y_9}^* \mathcal{I}_{x_2}(-t+1))$$

By applying the Grothendieck-Verdier duality 2.1.2 to the embedding $\iota_{Y_9}: Y_9 \hookrightarrow \widetilde{\mathcal{Y}}$, we have

$$\mathrm{Ext}_{Y_9}^{\bullet-1}(\mathcal{I}_{x_1;9}, \iota_{Y_9}^* \mathcal{I}_{x_2}(-t+1)) \simeq \mathrm{Ext}_{Y_9}^{\bullet}(\iota_{Y_9*} \mathcal{I}_{x_1;9}, \mathcal{I}_{x_2}(-t)).$$

Therefore we have

$$(9.5) \quad \mathrm{Ext}_{Y_9}^{\bullet-1}(\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2;9}(-t+1)) \simeq \mathrm{Ext}_{Y_9}^{\bullet}(\iota_{Y_9*} \mathcal{I}_{x_1;9}, \mathcal{I}_{x_2}(-t)).$$

Taking $\mathrm{Hom}(-, \mathcal{I}_{x_1}(-t))$ of the exact sequence

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}}(-1) \rightarrow \mathcal{I}_{x_1} \rightarrow \iota_{Y_9*} \mathcal{I}_{x_1;9} \rightarrow 0$$

(cf. Lemma 9.0.6 (2)), we obtain the exact sequence

$$(9.6) \quad H^{\bullet-1}(\mathcal{I}_{x_1}((-t+1))) \rightarrow \mathrm{Ext}_{Y_9}^{\bullet}(\iota_{Y_9*} \mathcal{I}_{x_1;9}, \mathcal{I}_{x_2}(-t)) \rightarrow \mathrm{Ext}_{\widetilde{\mathcal{Y}}}^{\bullet}(\mathcal{I}_{x_1}, \mathcal{I}_{x_2}(-t)),$$

where the last term vanishes from Proposition 9.0.5.

Claim 9.0.7. $H^{\bullet-1}(\mathcal{I}_{x_1}((-t+1))) = 0$.

Proof. The assertion follows from Proposition 8.1.1 and Theorem 9.0.4. \square

Therefore we have (9.3) from (9.5) and (9.6).

Step 2 (from Y_9 to Y_8).

In this step, we show

$$(9.7) \quad \mathrm{Ext}_{Y_8}^{\bullet-1}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t+1)) = 0 \quad (1 \leq t \leq 9).$$

Since $\mathcal{I}_{x_1;8} \simeq \iota_{Y_8}^* \mathcal{I}_{x_1;9}$, we have

$$(9.8) \quad \begin{aligned} \mathrm{Ext}_{Y_8}^{\bullet-1}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t+1)) &\simeq \mathrm{Ext}_{Y_8}^{\bullet-1}(\iota_{Y_8}^* \mathcal{I}_{x_1;9}, \mathcal{I}_{x_2;8}(-t+1)) \\ &\simeq \mathrm{Ext}_{Y_9}^{\bullet-1}(\mathcal{I}_{x_1;9}, \iota_{Y_8*} \mathcal{I}_{x_2;8}(-t+1)). \end{aligned}$$

From (9.8) and $\mathrm{Hom}(\mathcal{I}_{x_1;9}(t-1), -)$ of the exact sequence

$$0 \rightarrow \mathcal{O}_{Y_9}(-1) \rightarrow \mathcal{I}_{x_2;9} \rightarrow \iota_{Y_8*} \mathcal{I}_{x_2;8} \rightarrow 0$$

(cf. Lemma 9.0.6 (3)) we obtain the exact sequence

$$(9.9) \quad \begin{aligned} \mathrm{Ext}_{Y_9}^{\bullet-1}(\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2;9}(-t+1)) &\rightarrow \mathrm{Ext}_{Y_8}^{\bullet-1}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t+1)) \\ &\rightarrow \mathrm{Ext}_{Y_9}^{\bullet}(\mathcal{I}_{x_1;9}, \mathcal{O}_{Y_9}(-t)), \end{aligned}$$

where the first term vanishes from (9.3).

Claim 9.0.8. $\mathrm{Ext}_{Y_9}^{\bullet}(\mathcal{I}_{x_1;9}, \mathcal{O}_{Y_9}(-t)) = 0$.

Proof. Set $F_{Y_9} := F_{\widetilde{\mathcal{Y}}}|_{Y_9}$. By the Serre-Grothendieck duality, it holds

$$(9.10) \quad \mathrm{Ext}_{Y_9}^\bullet(\mathcal{I}_{x_1;9}, \mathcal{O}_{Y_9}(-t)) \simeq H^{12-\bullet}(Y_9, \mathcal{I}_{x_1;9}((t-9)M_{Y_9} + 2F_{Y_9}))^*$$

since $K_{Y_9} = -9M_{Y_9} + 2F_{Y_9}$. We show that

$$(9.11) \quad H^{12-\bullet}(Y_9, \mathcal{I}_{x_1;9}((t-9)M_{Y_9} + 2F_{Y_9})) \simeq H^{12-\bullet}(\widetilde{\mathcal{Y}}, \mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}))$$

Since $\Delta_{x_1;9} = \Delta_{x_1}$, we have the following two exact sequences on $\widetilde{\mathcal{Y}}$ and Y_9 respectively:

$$(9.12) \quad \begin{aligned} 0 \rightarrow \mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}) &\rightarrow \mathcal{O}_{\widetilde{\mathcal{Y}}}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}) \\ &\rightarrow \mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}) \rightarrow 0, \end{aligned}$$

and

$$(9.13) \quad \begin{aligned} 0 \rightarrow \mathcal{I}_{x_1;9}((t-9)M_{Y_9} + 2F_{Y_9}) &\rightarrow \mathcal{O}_{Y_9}((t-9)M_{Y_9} + 2F_{Y_9}) \\ &\rightarrow \mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}) \rightarrow 0. \end{aligned}$$

Since $(t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}} = (t+1)M_{\widetilde{\mathcal{Y}}} + K_{\widetilde{\mathcal{Y}}}$ and $(t+1)M_{\widetilde{\mathcal{Y}}}$ is nef and big, the cohomology groups $H^{12-\bullet}(\widetilde{\mathcal{Y}}, \mathcal{O}_{\widetilde{\mathcal{Y}}}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}))$ vanish for $12-\bullet > 0$ by the Kawamata-Viewheg vanishing theorem. Moreover, it holds that $H^0(\widetilde{\mathcal{Y}}, \mathcal{O}_{\widetilde{\mathcal{Y}}}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}})) \simeq H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}((t-9)M_{\mathcal{Y}}))$, and the latter vanishes if $t \leq 8$, and is isomorphic to \mathbb{C} if $t = 9$. If $t = 9$, then $H^0(\widetilde{\mathcal{Y}}, \mathcal{O}_{\widetilde{\mathcal{Y}}}(2F_{\widetilde{\mathcal{Y}}})) \simeq H^0(\Delta_{x_1}, \mathcal{O}_{\Delta_{x_1}}(2F_{\widetilde{\mathcal{Y}}}))$ and hence $H^0(\widetilde{\mathcal{Y}}, \mathcal{I}_{x_1}(2F_{\widetilde{\mathcal{Y}}})) = 0$ by (9.12). In the remaining cases, we have $H^{12-\bullet}(\widetilde{\mathcal{Y}}, \mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}})) \simeq H^{11-\bullet}(\Delta_{x_1}, \mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}))$ by (9.12). Similarly, by (9.13) and the Kawamata-Viewheg vanishing theorem, we have $H^0(Y_9, \mathcal{I}_{x_1;9}(2F_{Y_9})) = 0$, and in the remaining cases, $H^{12-\bullet}(Y_9, \mathcal{I}_{x_1;9}((t-9)M_{Y_9} + 2F_{Y_9})) \simeq H^{11-\bullet}(\Delta_{x_1}, \mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}))$, where we need $Y_9 = \mathcal{V}_{x_1}$ has only canonical singularities (see Proposition 7.0.2).

Therefore we have the isomorphism (9.11).

Now we have only to show the vanishings of $H^{12-\bullet}(\widetilde{\mathcal{Y}}, \mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}))$ by (9.10) and (9.11). These follow from the vanishings of

$$\begin{aligned} &H^{12-\bullet}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{S}}_L^*((t-10)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}})), \quad H^{12-\bullet}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{T}}^*((t-10)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}})), \\ &H^{12-\bullet}(\widetilde{\mathcal{Y}}, \mathcal{O}_{\widetilde{\mathcal{Y}}}((t-10)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}})), \quad H^{12-\bullet}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Q}}^*((t-9)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}})) \end{aligned}$$

by (8.1). Those cohomology groups are Serre-dual to

$$\begin{aligned} &H^{\bullet+1}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{S}}_L(-t)), \quad H^{\bullet+1}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{T}}(-t)), \\ &H^{\bullet+1}(\widetilde{\mathcal{Y}}, \mathcal{O}_{\widetilde{\mathcal{Y}}}(-t)), \quad H^{\bullet+1}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Q}}(-t-1)), \end{aligned}$$

which vanish by Theorem 9.0.4. Therefore, by (9.10), we have $\mathrm{Ext}_{Y_9}^\bullet(\mathcal{I}_{x_1;9}, \mathcal{O}_{Y_9}(-t)) = 0$. \square

Now (9.7) follows from (9.9).

Step 3 (from Y_8 to Y_7).

In this step, we show

$$(9.14) \quad \mathrm{Ext}_{Y_7}^{\bullet-1}(\mathcal{I}_{x_1;7}, \mathcal{I}_{x_2;7}(-t+1)) = 0 \quad (1 \leq t \leq 8).$$

Since $\mathcal{I}_{x_1;7} \simeq \iota_{Y_7}^* \mathcal{I}_{x_1;8}$, we have

$$(9.15) \quad \begin{aligned} \mathrm{Ext}_{Y_7}^{\bullet-1}(\mathcal{I}_{x_1;7}, \mathcal{I}_{x_2;7}(-t+1)) &\simeq \mathrm{Ext}_{Y_7}^{\bullet-1}(\iota_{Y_7}^* \mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;7}(-t+1)) \\ &\simeq \mathrm{Ext}_{Y_8}^{\bullet-1}(\mathcal{I}_{x_1;8}, \iota_{Y_7*} \mathcal{I}_{x_2;7}(-t+1)). \end{aligned}$$

Note that a defining equation of Y_7 restricts to a regular element in each local ring of $\Delta_{x_2;8}$. Hence the sequence

$$(9.16) \quad 0 \rightarrow \mathcal{I}_{x_2;8}(-1) \rightarrow \mathcal{I}_{x_2;8} \rightarrow \iota_{Y_7*} \mathcal{I}_{x_2;7} \rightarrow 0$$

is exact. Therefore, by (9.15), and $\mathrm{Hom}(\mathcal{I}_{x_1;8}(t-1), -)$ of (9.16), we have the following exact sequence:

$$(9.17) \quad \begin{aligned} \mathrm{Ext}_{Y_8}^{\bullet-1}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t+1)) &\rightarrow \mathrm{Ext}_{Y_7}^{\bullet-1}(\mathcal{I}_{x_1;7}, \mathcal{I}_{x_2;7}(-t+1)) \\ &\rightarrow \mathrm{Ext}_{Y_8}^{\bullet}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t)), \end{aligned}$$

where $\mathrm{Ext}_{Y_8}^{\bullet-1}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t+1))$ and $\mathrm{Ext}_{Y_8}^{\bullet}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t))$ vanish by (9.7) if $1 \leq t \leq 8$, and then we have (9.14).

Step 4 (from Y_7 to Y_6, \dots, Y_1 to Y).

In this step, we finish the proof of (9.2). Since a defining equation of Y_i restricts to a regular element in each local ring of both $\Delta_{x_1;i+1}$ and $\Delta_{x_2;i+1}$, we can show inductively the following vanishing for any $i \in [0, 6]$ in a similar way to the argument of Step 3:

$$(9.18) \quad \mathrm{Ext}_{Y_i}^{\bullet-1}(\mathcal{I}_{x_1;i}, \mathcal{I}_{x_2;i}(-t+1)) = 0 \quad (1 \leq t \leq i+1).$$

In particular, we have $\mathrm{Ext}_Y^{\bullet-1}(I_{x_1}, I_{x_2}) = 0$, which is (9.2). \square

10. FURTHER DISCUSSION

In our proof of the derived equivalence, we have used (the ideal sheaf of) a family of curves $\{C_x\}_{x \in X}$ which arises from the restriction $\mathcal{C} = \Delta|_{X \times Y}$. Obviously, the other choice of a family $\{C_y\}_{y \in Y}$ should be possible for that purpose. In this section, we obtain a flat family of curves on X from Δ for the latter choice, and remark, however, that a technical problem prevent us to complete a proof by using this family.

We also make a comment on non-invariances of the fundamental groups and the Brauer groups under the derived equivalence.

10.1. A family of curves on X .

For a point $y \in \widetilde{\mathcal{Y}}$, we denote by Q_y the quadric in \mathbb{P}^4 corresponding to the image of y on \mathcal{H} . We also denote by Δ_y the fiber of $\Delta \rightarrow \widetilde{\mathcal{Y}}$ over y , which is a closed subscheme of \mathcal{X} .

Let $V_{\widetilde{\mathcal{Y}}}$ be the open subset of $\widetilde{\mathcal{Y}}$ consisting of points y such that $\mathrm{rank} Q_y = 3$ or 4. For a point $y \in V_{\widetilde{\mathcal{Y}}}$, let $q_y \subset G(3, V)$ be the conic corresponding to y (q_y is one of the connected components of the families of planes in Q_y). We describe Δ_y for $y \in V_{\widetilde{\mathcal{Y}}}$.

Proposition 10.1.1. Δ_y is the restriction of $\tilde{\mathcal{X}} = \mathbb{P}(\mathcal{S}^2 \mathcal{F}^*) \rightarrow \mathbf{G}(2, V)$ over the subset

$$G_y := \{l \mid l \text{ is contained in a plane belonging to } q_y\} \subset \mathbf{G}(2, V).$$

If $\text{rank } Q_y = 4$, then G_y is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ and $\mathcal{O}_{\mathbf{G}(2, V)}(1)|_{G_y}$ is the tautological divisor. If $\text{rank } Q_y = 3$, then G_y is isomorphic to the image of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2})$ by the tautological linear system and $\mathcal{O}_{\mathbf{G}(2, V)}(1)|_{G_y}$ is

In particular, Δ_y is smooth if $\text{rank } Q_y = 4$.

Proof. The first part easily follows from the definition of Δ .

We describe G_y . By definition, G_y is nothing but the restriction of $\Delta_0 \rightarrow \mathbf{G}(3, V)$ over q_y . Since $\Delta_0 = \mathbf{F}(2, 3, V)$, we have $\Delta_0 = \mathbb{P}(\mathcal{U})$ as a \mathbb{P}^2 -bundle over $\mathbf{G}(3, V)$. We denote by p_1 the projection $\Delta_0 \rightarrow \mathbf{G}(3, V)$ and by p_2 the projection $\Delta_0 \rightarrow \mathbf{G}(2, V)$. We show

$$(10.1) \quad \mathcal{O}_{\mathbb{P}(\mathcal{U})}(1) \simeq p_2^* \mathcal{O}_{\mathbf{G}(2, V)}(1) \otimes p_1^* \mathcal{O}_{\mathbf{G}(3, V)}(-1).$$

Indeed, by $\mathcal{U} \simeq \wedge^2 \mathcal{U}^* \otimes \mathcal{O}_{\mathbf{G}(3, V)}(1)$, we have $\mathbb{P}(\mathcal{U}) \simeq \mathbb{P}(\wedge^2 \mathcal{U}^*)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{U})}(1) \simeq \mathcal{O}_{\mathbb{P}(\wedge^2 \mathcal{U}^*)}(1) \otimes p_1^* \mathcal{O}_{\mathbf{G}(3, V)}(-1)$. Moreover, by the universal exact sequence $0 \rightarrow \mathcal{U}^* \rightarrow V \otimes \mathcal{O}_{\mathbf{G}(3, V)} \rightarrow \mathcal{W} \rightarrow 0$, we obtain the injection $\wedge^2 \mathcal{U}^* \rightarrow \wedge^2 V \otimes \mathcal{O}_{\mathbf{G}(3, V)}$. Therefore, $\mathcal{O}_{\mathbb{P}(\wedge^2 \mathcal{U}^*)}(1) = p_2^* \mathcal{O}_{\mathbf{G}(2, V)}(1)$, which implies (10.1). Note that (10.1) is equivalent to $\mathcal{O}_{\mathbb{P}(\mathcal{U}(-1))}(1) \simeq p_2^* \mathcal{O}_{\mathbf{G}(2, V)}(1)$.

By the discussion in the beginning of [HoTa3, Subsect.5.3], $\mathcal{U}(-1)|_{q_y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ if $\text{rank } Q_y = 4$, and $\mathcal{U}(-1)|_{q_y} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2}$ if $\text{rank } Q_y = 3$. Therefore, G_y is as in the statement. \square

Similarly to the proof of Proposition 8.1.1, we can prove the following:

Proposition 10.1.2. The scheme Δ is flat over $V_{\tilde{\mathcal{Y}}}$. The ideal sheaf \mathcal{I}_y of Δ_y in $\tilde{\mathcal{X}}$ is $\mathcal{I} \otimes \mathcal{O}_{\tilde{\mathcal{X}}_y}$ for $y \in V_{\tilde{\mathcal{Y}}}$, where $\tilde{\mathcal{X}}_y \simeq \tilde{\mathcal{X}}$ is the fiber of $\tilde{\mathcal{Y}} \times \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$ over y . Moreover, the exact sequence (6.2) remains to be exact after restricting on $\tilde{\mathcal{X}}_y$ and gives the following locally free resolution of \mathcal{I}_y :

$$(10.2) \quad \begin{aligned} 0 \rightarrow g^* \mathcal{O}_{\mathbf{G}(2, V)}(-2)^{\oplus 3} \rightarrow g^* (\mathcal{F}(-2))^{\oplus 4} \rightarrow \\ g^* \mathcal{S}^2 \mathcal{F}^* \oplus g^* \mathcal{O}_{\mathbf{G}(2, V)}(-1)^{\oplus 3} \rightarrow \mathcal{I}_y \rightarrow 0. \end{aligned}$$

Proof. By Proposition 10.1.1, $\Delta \rightarrow \tilde{\mathcal{Y}}$ is equidimensional over $V_{\tilde{\mathcal{Y}}}$. Therefore all the assertions can be proved in the same way of the proof of Proposition 8.1.1. \square

The following computation is standard:

Lemma 10.1.3. $c_1(\mathcal{S}^2 \mathcal{F}) = c_1(\mathcal{O}_{\mathbf{G}(2, V)}(3))$, $c_2(\mathcal{S}^2 \mathcal{F}) = 2c_1(\mathcal{O}_{\mathbf{G}(2, V)}(1))^2 + 4c_2(\mathcal{F})$, and $c_3(\mathcal{S}^2 \mathcal{F}) = 4c_1(\mathcal{O}_{\mathbf{G}(2, V)}(1))c_2(\mathcal{F})$.

Now we can derive the following:

Proposition 10.1.4. The scheme \mathcal{C} defined as in Subsection 8.2 is flat over Y . Let C_y be the fiber of \mathcal{C} over a point $y \in Y$. Then C_y is a curve of arithmetic genus 14 and degree 20. Moreover, if X and Y are general, then a general C_y is smooth.

Proof. Note that $Y \subset V_{\mathcal{Y}}$. Take a point $y \in Y$. We only consider the case of $\text{rank } Q_y = 4$ since the other case can be studied similarly.

First we show that $\deg \Delta_y = 20$ with respect to $H_{\mathcal{X}} = H_{\mathbb{P}(\mathcal{S}^2 \mathcal{F})}$. The degree of Δ_y is evaluated by the Segre class of $\mathcal{S}^2 \mathcal{F}$ as $s_3(\mathcal{S}^2 \mathcal{F})G_y$. By the formula $s_3(\mathcal{S}^2 \mathcal{F}) = c_3(\mathcal{S}^2 \mathcal{F}) - 2c_1(\mathcal{S}^2 \mathcal{F})c_2(\mathcal{S}^2 \mathcal{F}) + c_1(\mathcal{S}^2 \mathcal{F})^3$ and Lemma 10.1.3, we have $\deg \Delta_y = (15c_1(\mathcal{O}_{G(2,V)}(1)))^3 - 20c_1(\mathcal{O}_{G(2,V)}(1))c_2(\mathcal{F})G_y$. By Proposition 10.1.1, we have $\deg G_y = c_1(\mathcal{O}_{G(2,V)}(1))^3 G_y = 4$. Note that $c_2(\mathcal{F}) = [\mathbf{G}(2, V_4)]$ as codimension 2 cycle with some 4-dimensional space $V_4 \subset V$. Then we see that $c_2(\mathcal{F})G_y$ is represented by a conic, which parameterizes a \mathbb{P}^1 -family of rulings in the smooth quadric surface $Q_y \cap \mathbb{P}(V_4)$. Therefore we have $c_1(\mathcal{O}_{G(2,V)}(1))c_2(\mathcal{F})G_y = 2$. Consequently, we have $\deg \Delta_y = 20$.

We show that C_y is a curve. Similarly to the proof of Proposition 8.2.1, C_y is a complete intersection in Δ_y by 4 members of $|H_{\mathcal{X}}|$. Therefore $\dim C_y \geq 1$. Assume that $\dim C_y = 2$. Then $\deg C_y \leq 20$ since $\deg \Delta_y = 20$. This is, however, impossible since $H_{\mathcal{X}}|_X$ generates $\text{Pic } X$ modulo torsion and $(H_{\mathcal{X}}|_X)^3 = 35$. Therefore C_y is a curve, and then \mathcal{C} is flat over Y as in the proof of Proposition 8.2.1.

Now we compute the canonical divisor of C_y . Since $\Delta_y \rightarrow G_y$ is a projective bundle, we have $K_{\Delta_y} = -3H_{\mathcal{X}}|_{\Delta_y} + (g|_{\Delta_y})^*(c_1(\mathcal{S}^2 \mathcal{F})|_{G_y} + K_{G_y})$. Since we have seen that C_y is a complete intersection in Δ_y by 4 members of $|H_{\mathcal{X}}|$, we have $K_{C_y} = H_{\mathcal{X}}|_{C_y} + (g|_{\Delta_y})^*(c_1(\mathcal{S}^2 \mathcal{F})|_{G_y} + K_{G_y})|_{C_y}$. Therefore $\deg K_{C_y} = H_{\mathcal{X}}^5 \Delta_y + H_{\mathcal{X}}^4 (g|_{\Delta_y})^*(c_1(\mathcal{S}^2 \mathcal{F})|_{G_y} + K_{G_y})$. We have already computed $H_{\mathcal{X}}^5 \Delta_y = 20$. By using the Segre class of $\mathcal{S}^2 \mathcal{F}$, we have

$$H_{\mathcal{X}}^4 (g|_{\Delta_y})^*(c_1(\mathcal{S}^2 \mathcal{F})|_{G_y} + K_{G_y}) = s_2(\mathcal{S}^2 \mathcal{F})|_{G_y} (c_1(\mathcal{S}^2 \mathcal{F})|_{G_y} + K_{G_y}).$$

By Lemma 10.1.3, we have $s_2(\mathcal{S}^2 \mathcal{F}) = c_1(\mathcal{S}^2 \mathcal{F})^2 - c_2(\mathcal{S}^2 \mathcal{F}) = 7c_1(\mathcal{O}_{G(2,V)}(1))^2 - 4c_2(\mathcal{F})$. By Proposition 10.1.1, we have $\mathcal{O}_{G_y}(K_{G_y}) \simeq \mathcal{O}_{G(2,V)}(-3)|_{G_y} \otimes p^* \mathcal{O}_{\mathbb{P}^1}(2)$, where p is the natural morphism $p: G_y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \rightarrow q_y \simeq \mathbb{P}^1$. Thus $c_1(\mathcal{S}^2 \mathcal{F})|_{G_y} + K_{G_y} = p^* \mathcal{O}_{\mathbb{P}^1}(2)$. Hence

$$H_{\mathcal{X}}^4 (g|_{\Delta_y})^*(c_1(\mathcal{S}^2 \mathcal{F})|_{G_y} + K_{G_y}) = (7c_1(\mathcal{O}_{G(2,V)}(1))^2 - 4c_2(\mathcal{F}))|_{G_y} p^* \mathcal{O}_{\mathbb{P}^1}(2).$$

Recall that $c_2(\mathcal{F})|_{G_y}$ is represented as a conic on $\mathbf{G}(2, V)$, which is a section of $p: G_y \rightarrow q_y$. Therefore $(7c_1(\mathcal{O}_{G(2,V)}(1))^2 - 4c_2(\mathcal{F}))|_{G_y} p^* \mathcal{O}_{\mathbb{P}^1}(2) = 6$. Consequently, we have $\deg K_{C_y} = 26$, equivalently, $p_a(C_y) = 14$.

Let y be a point of \mathcal{Y} such that $\text{rank } Q_y = 4$. Take a general 4-plane P in $\mathbb{P}(\mathcal{S}^2 V^*)$ containing $[Q_y]$ and define X and Y as before. Note that $y \in Y$. Write $P = \langle Q_y, Q_1, \dots, Q_4 \rangle$. Let H_i be the member of $|H_{\mathcal{X}}|$ corresponding to Q_i . Note that $\Delta_y \subset \mathcal{V}_y$ by Proposition 7.0.1. Since P is general, H_i are general members of $|H_{\mathcal{X}}|$. Therefore, C_y is smooth since so is Δ_y and C_y is a complete intersection in Δ_y by H_1, \dots, H_4 . \square

One might consider that we can show the derived equivalence by the Fourier-Mukai functor with the ideal sheaves of the family $\{C_y\}_{y \in Y}$. Actually, computations of Ext groups among the sheaves appearing in the resolution (10.2) are easier than Theorem 9.0.4 (see [HoTa3, Thm.3.4.4]). However, one technical problem is involved as follows: When we follow the argument of Section 9, it is crucial to have the property that, for *any* distinct two points y_1 and y_2 of Y , the hyperplane sections \mathcal{V}_{y_1} and \mathcal{V}_{y_2} of X are different (cf. Lemma 9.0.6). This, however, does not hold for y_1 and y_2 such that their images on H coincides. This forced us to choose $\{C_x\}_{x \in X}$ in our proof.

Remark. It should be interesting to find the curves C_y ($y \in Y$) of genus 14 and degree 20 in the table of the BPS numbers of X [HoTa1, Table 2]. In fact, we read from the table the counting number of the curves of genus 14 and degree 20 as

$$n_{14}^X(20) = 500.$$

In a similar way to the BPS number $n_3^Y(5) = 100$ discussed in Introduction, we may arrange this number as $n_{14}^X(20) = (-1)^{\dim Y} e(Y) \times 10$. This time, however, it is not clear whether the factor 10 has a nice interpretation from the geometry of X . Nevertheless, we expect that the number $n_{14}^X(20) = 500$ is 'counting' the Euler numbers of the parameter spaces of generically smooth family of curves on X by general properties of the BPS numbers [GV], since we were able to verify $n_{14}^X(21) = 0$ after heavy calculations using mirror symmetry.

Remark. In the Grassmann-Pfaffian case due to [BC, Ku2], the constructions of curves $\{C_x\}_{x \in X}$ and $\{C_y\}_{y \in Y}$ are more straightforward. Assume X is a smooth linear section Calabi-Yau threefold of $G(2, 7)$, which is embedded in $\mathbb{P}(\wedge^2 \mathbb{C}^7)$, and Y is the corresponding (smooth) orthogonal linear section Calabi-Yau threefolds of $\text{Pf}(7)$ in $\mathbb{P}(\wedge^2 (\mathbb{C}^*)^7)$. Let us write by $[\xi_x] = [\xi_x^{(1)}, \xi_x^{(2)}] \simeq \mathbb{P}^1$ the line corresponding to a point $x \in G(2, 7)$, and by $[\eta_y]$ a skew symmetric matrix rank $\eta_y \leq 4$ corresponding to $y \in \text{Pf}(7)$. Then the incidence relation used in [BC, Ku2] is $\Delta = \{([\xi_x], [\eta_y]) \mid \dim(\xi_x \cap \text{Ker}(\eta_y)) \geq 1\}$. In this case, the fiber Δ_y of $\Delta \rightarrow \text{Pf}(7)$ over y is given by a Schubert cycle σ_3 in $G(2, 7)$ of codimension 3 if $\text{rank}(\eta_y) = 4$, and simplifies the proof of the derived equivalence using the family of curves $\{C_y\}_{y \in Y} = \{\Delta \cap (X \times \{y\})\}_{y \in Y}$. As discussed in Introduction, C_y is generically a smooth curve on X of genus 6 and degree 14. The fiber Δ_x of the other fibration $\Delta \rightarrow G(2, 7)$ is easy to be described. It turns out that

$$\Delta_x = \{([\xi_x], [\eta_y]) \mid (\eta_y \xi_x^{(1)}) \wedge (\eta_y \xi_x^{(2)}) = 0\}.$$

When X and Y are generic and smooth, we verified by *Macaulay2* that $C_x = \Delta \cap (\{x\} \times Y)$ ($x \in X$) is generically a smooth curve on Y of genus 11 and degree 14. The corresponding BPS number is, unfortunately, outside of the tables available in literatures (see [HoTa1, Section 4]).

10.2. The fundamental groups and the Brauer groups of X and Y .

Finally, it should be worth while discussing about non-invariance of the fundamental groups and the Brauer groups by the derived equivalence between a Reye congruence X and the double symmetroid Y orthogonal to X .

As for the fundamental groups, we have $\pi_1(X) \simeq \mathbb{Z}_2$ by [HoTa3, Prop.3.5.3], and $\pi_1(Y) \simeq 0$ by [ibid. Prop.4.3.4]. To our best knowledge, this seems to be the second example of pairs of derived equivalent Calabi-Yau threefolds with different fundamental groups (see [S]).

As for the Brauer groups, we follow the argument of [A]: By [BK], the Atiyah-Hirzebruch spectral sequence gives a short exact sequence for any Calabi-Yau threefold Σ :

$$(10.3) \quad 0 \rightarrow H_1(\Sigma, \mathbb{Z}) \rightarrow K_{\text{top}}^1(\Sigma)_{\text{tors}} \rightarrow \text{Br}(\Sigma) \rightarrow 0,$$

where $K_{\text{top}}(\Sigma) = K_{\text{top}}^0(\Sigma) \oplus K_{\text{top}}^1(\Sigma)$ is the topological K -group and the subscript 'tors' means the torsion part. As we mentioned above, we have $H_1(X, \mathbb{Z}) \simeq \pi_1(X) \simeq$

\mathbb{Z}_2 and $H_1(Y, \mathbb{Z}) \simeq 0$. By [AT, §2.2], $K_{\text{top}}^1(X) \simeq K_{\text{top}}^1(Y)$ since X and Y are derived equivalent. Therefore, by (10.3), we have the following relation between the Brauer groups of X and Y :

$$(10.4) \quad 0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Br}(Y) \rightarrow \text{Br}(X) \rightarrow 0.$$

We have shown $\text{Br}(Y)$ contains a nonzero 2-torsion element in Proposition 3.2.1. If $\text{Br}(Y) \simeq \mathbb{Z}_2$, we will have $\text{Br}(X) \simeq 0$ by (10.4), which indicates that the Brauer groups are not invariant under the derived equivalence.

REFERENCES

- [A] N. Addington, *The Brauer group is not a derived invariant*, preprint, available at <http://math.duke.edu/~adding/>
- [AT] N. Addington, and R. P. Thomas, *Hodge theory and derived categories of cubic fourfolds*, arXiv:1211.3758.
- [BK] V. Batyrev, and M. Kreuzer, *Integral cohomology and mirror symmetry for Calabi-Yau 3-folds*, Mirror symmetry. V, 255–270, AMS/IP Stud. Adv. Math., 38, Amer. Math. Soc., Providence, RI, 2006.
- [BC] L. Borisov and A. Caldararu, *The Pfaffian-Grassmannian derived equivalence*, J. Algebraic Geom. **18** (2009), no. 2, 201–222.
- [BO] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, arXiv:alg-geom/9506012
- [Bo] R. Bott, *Homogeneous vector bundles*, Ann. of Math. (2) **66** (1957), 203–248
- [B] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. **31** (1999), no. 1, 25–34.
- [D] M. Demazure, *A very simple proof of Bott’s theorem*, Invent. Math. **33** (1976), no. 3, 271–272.
- [Ful] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **2**, Springer-Verlag, Berlin, 1984. xi+470 pp.
- [FH] W. Fulton and J. Harris, *Representation Theory, a first course*, GTM 129, Springer-Verlag 1991.
- [GV] R. Gopakumar and C. Vafa, *M-Theory and Topological Strings–II*, hep-th/9812127.
- [H] R. Hartshorne, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, 1961. Lecture Notes in Mathematics, No. **41** Springer-Verlag, Berlin-New York 1967 vi+106 pp.
- [HoTa1] S. Hosono and H. Takagi, *Mirror symmetry and projective geometry of Reye congruences I*, arXiv:1101.2746, to appear in Journal of Algebraic Geometry.
- [HoTa2] ———, *Determinantal quintics and mirror symmetry of Reye congruences*, preprint, arXiv:1208.1813.
- [HoTa3] ———, *Duality between Chow² \mathbb{P}^4 and the double symmetric determinantal quintic*, preprint.
- [Huy] D. Huybrechts, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford Mathematical Monographs, Oxford 2006.
- [Ko] M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians (Zürich, 1994) Birkhäuser (1995) pp. 120–139.
- [IM] A. Iliev and L. Manivel, *Fano manifolds of degree ten and EPW sextics*, Annales scientifiques de l’Ecole Normale Supérieure **44** (2011), 393–426.
- [Ku1] A. Kuznetsov, *Homological projective duality*, Publ. Math. Inst. Hautes Études Sci. No. 105 (2007), 157–220.
- [Ku2] ———, *Homological projective duality for Grassmannians of lines*, arXiv:math/0610957.

- [M] H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, **8**. Cambridge University Press, Cambridge, 1989. xiv+320
- [PT] R. Pandharipande and R.P. Thomas, *Stable pairs and BPS invariants*. J. Amer. Math. Soc. **23** (2010), no. 1, 267–297.
- [Ro] E.A. Rødland, *The Pfaffian Calabi-Yau, its Mirror and their link to the Grassmannian $G(2, 7)$* , Compositio Math. **122** (2000), no. 2, 135–149.
- [S] C. Schnell, *The fundamental group is not a derived invariant*, arXiv:1112.3586.
- [W] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, **149**. Cambridge University Press, Cambridge, 2003. xiv+371 pp.

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